A FIRST COURSE

IN

INFINITESIMAL CALCULUS

W. W. Beman

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BY

DANIEL A. MURRAY, PH.D. (JOHNS HOPKINS)
PROFESSOR OF MATHEMATICS IN DALHOUSIE COLLEGE,
HALIFAX, N.S.

NEW EDITION

LONGMANS, GREEN, AND CO.
91 AND 93 FIFTH AVENUE, NEW YORK
LONDON AND BOMBAY
1904

First Edition, September, 1903.
Reprinted May, 1904.

Norwood Press
J. S. Cushing & Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

Mrs. Beman
pt.
4-30-1924

PREFACE.

THIS book has been written for beginners in calculus. Its purpose is to provide an introductory course for those who are entering upon that study either to prepare themselves for elementary work in applied science or to gratify and develop their interest in mathematics. This purpose has determined the choice and the arrangement of the topics and the mode of presentation. Little more has been discussed than what may be regarded as the essentials of a primary course in calculus. An attempt is made to describe and emphasise the fundamental principles of the subject in such a way that, as much as may reasonably be expected, they may be clearly understood, firmly grasped, and intelligently applied by young students. There has also been kept in view the development in them of the ability to read mathematics and to prosecute its study by themselves.

Excepting in a few instances, only real functions of real variables are considered. Simple, practical applications of the more elementary notions are introduced as early as possible; and, subject to the requirements of a logically connected development of the study, the more difficult and abstract discussions appear later. In accordance with this plan, the time-honoured division into differential calculus and integral calculus has not been made, and, to mention one instance in particular, following the example set by Professors Lamb, Gibson, and others, the development of functions in series is taken up in the latter, instead of in the earlier, part of the course. The book, however, can be divided easily into differential and integral sections, and thus can be adapted, in this respect at least, for use in cases in which such a division is deemed necessary.

With regard to simplicity and clearness in the exposition of the subject, it may be said that the aim has been to write a book

that will be found helpful by those who begin the study of calculus without the guidance and aid of a teacher. For these students more especially, throughout the work suggestions and remarks are made concerning the order in which the various topics may be studied, the relative importance of the various topics in a first study of calculus, the articles that must be thoroughly mastered, and the articles that may advantageously be omitted or lightly passed over at the first reading, and so on.

The notion of anti-differentiation is presented simultaneously with the notion of differentiation, and exercises thereon appear early in the text; but in the chapter in which integration is formally taken up the idea of integration as a process of summation is considered before the idea of integration as a process which is the inverse of differentiation. In this matter I have followed the order adopted in my Integral Calculus, although there is considerable difference of opinion as to the propriety or the advantage of this order. The decision to follow it here has been made mainly fòr the reason that students appear — at least so it seems to me, but other teachers may have a different experience — to understand more clearly and vividly the relation of integration to many practical problems when the summation idea is put in the forefront. In teaching the one order can be taken as readily as the other.

In several technical schools the time assigned to calculus is not sufficient for a fair study of Taylor's theorem. What may be regarded as the irreducible requisite for a slight working acquaintance with Taylor's and Maclaurin's series is indicated at the beginning of Chapter XIX., and may be taken at an early stage in the course.

The evaluation of indeterminate forms, which affords interesting exercises in the application of differentiation, is far from being as important as many other applications of the calculus; and in the few cases in which this evaluation is required it can be effected by other means. Useful exercises in applying integration can be given to students who have a knowledge of mechanics. In many cases, however, these students make but a small fraction of the class, and, besides, in a large number of technical schools the curriculum provides that mechanics shall

follow calculus. Accordingly, it seemed better not to treat indeterminate forms and mechanics in the body of the text, but to deal briefly with them in the appendix.

An explanation of hyperbolic functions can be made more naturally and more fully, perhaps, in a course in calculus than in any other course in elementary mathematics. For this reason, and also because students will meet them in their later work and reading, a note on these functions appears in the latter part of the book.

Owing to the pressure of other subjects the time allotted to mathematics in quite a number of technical schools is rather brief. Where this is the case, and where there is a lack of maturity in the students, it is better not to try to cover too much ground, but to lay stress on fundamental principles, to drill in the elementary processes, and to train in making simple applications. Thus this book, small as it may be regarded even for a short course, contains more matter than can be thoroughly studied in the few months allotted to calculus in colleges and technical schools where such conditions exist. Several topics, however (for example, the investigation of series), which in some cases are not studied by technical students owing to lack of time, are very important, particularly for those who take a first course in the calculus as an introduction to a more extended study of the subject and as part of the preparation necessary for more advanced work in mathematics. For the sake of these students more especially, but not exclusively on their account, many definite references for collateral reading or inspection are given throughout the text.

It is hoped that these references will add to the helpfulness of the book. With but very few exceptions those are chosen which are easily accessible to all college students. Some of the references will aid the learner by presenting an idea of the text in the words of another; but the larger number of them are intended to direct students to places where they will either receive fuller information or be impressed with some of the important modern ideas of mathematics. Turning up such references as these will increase the mathematical interest of the student and widen his outlook. It will also help to train

the pupils in the use of mathematical literature, and, by arousing and exercising their critical faculties, will greatly benefit those who may intend to teach mathematics in the secondary schools. Of course the lists of references are not exhaustive, and, while care has been taken in making them, it is to be expected that several other equally serviceable lists can be arranged. It is intended that these lists shall be revised and supplemented by those who may use the book.

For learners who can afford but a minimum of time for this study the essential articles of a short course are indicated after the table of contents.

The exposition given here is, in the main, a result of my experience in teaching the calculus to a large number of pupils. Accordingly, it is my duty to acknowledge my indebtedness to many students whose difficulties and original opinions have interested and stimulated me. In preparing the text many works and articles have been consulted. I feel myself to be especially indebted to the writers to whom references are made in various places in the book.

Not many examples involving a technical knowledge of engineering, physics, or chemistry have been inserted. Few young students understand examples of this kind without considerable explanation, and thus it seems better to refer the pupils to the more specialised text-books dealing with calculus (for instance, those of Perry, Young and Linebarger, and Mellor), which contain many examples of a technical character.

I take this opportunity of thanking A. T. Bruegel, M.M.E., Professor of Mechanical Engineering in the Drexel Institute, Philadelphia, for advice and suggestions concerning the drawings, and Louis C. Loewenstein, Ph.D., Instructor in Mechanical Engineering in Lehigh University, for the interest and care taken by him in making the figures. I also wish to thank Miss A. A. Stewart, B.Sc., and my colleague, Professor H. Murray for kindly help in the revision of the proof-sheets. Miss Stewart also gave valuable assistance in verifying many of the examples.

D. A. MURRAY.

DALHOUSIE COLLEGE, HALIFAX, N.S.
August 15, 1903.

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SHORT COURSE

FOR STUDENTS HAVING A MINIMUM OF TIME

(The Roman numerals refer to chapters, the Arabic to articles.)

I.; II.; III. 17, 18, 21–27 *a*; IV.; V. 57–65; VI. 68–70; VII.; VIII. 79–84, 86; (IX.); X.; XI.; XII.; (XIII. 114–116, 118–122); XIV.; XV.; XVI. 134–139, 141; XIX. (167–171), 174; XX. 175, 180, Exs. 176–178; XVII.; XVIII.

INFINITESIMAL CALCULUS.

CHAPTER I.

INTRODUCTORY PROBLEMS.

1. The infinitesimal calculus is one of the most powerful mathematical instruments ever invented.* Many practical problems can be solved by its means with wonderful ease and rapidity. Even a slight acquaintance with the calculus is very helpful in the study of many other subjects, for example, geometry, astronomy, physics, and engineering; and the fullest knowledge possible about the calculus is necessary for advance in these subjects. Some of the higher branches of mathematics consist largely of special investigations in the infinitesimal calculus and extensions of its principles, methods, and applications.†

In this book the fundamental notions and principles of the calculus are, to a certain extent, explained, and applications are made to the solution of some simple practical problems. As a preliminary to the study there is in this chapter a discussion of a few problems. This discussion introduces in an informal way the notions and principles and methods which are at the foundation of the infinitesimal calculus, and also provides material which serves to illustrate a few of the articles that follow.‡

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* The calculus as used to-day was invented independently by Newton and Leibnitz. See Art. 94, note.

† The word "infinitesimal" serves to distinguish the subject from other branches of mathematics, such as the calculus of finite differences, the calculus of variations, the calculus of quaternions, etc.

‡ An important fact in the history of the calculus is that the problems in Arts. 3–6 were the occasion of the invention and development of some divisions of the subject.

Note. A knowledge of the meaning of the term *speed* or *rate of motion* is presupposed in the following two articles. If a body moves through equal distances in equal times, it is said to have *uniform* speed. The *average* speed of a body during the time that it is moving through a certain distance, is the uniform speed at which a body will pass over that distance in that time. For instance, if a bicyclist wheels 36 *miles* in 3 *hours*, his average speed is 12 *miles per hour*; if a body moves through 45 feet in 5 minutes, its average speed is 9 feet per minute. The *number* which indicates the average speed of a body while it is moving through a certain distance, is the ratio of the number of units of length in the distance to the number of units of time spent during the motion. In other words, the *measure of the speed* is the ratio of the measure of the distance to the measure of the corresponding time. Thus, in the instances above, $12 = 36 : 3$, $9 = 45 : 5$.

Any reader of this book knows what is meant by the statements that a train is running at a particular instant at the rate of 30 miles an hour, and that at another instant, some minutes later say, it is running at the rate of 40 miles an hour. This notion, viz. **the speed of a moving body at a particular instant,** will be developed further by the examples that follow.

2. Speed of a moving train. Suppose that a person is standing by a railway and wishes to ascertain the speed at which a train is going by him. A way to determine this speed approximately would be to find the distance passed over in five seconds by the train, or by a definite mark on the train, say a vertical line. (The place where the observer stands may be at one end of, or upon, the measured distance.) If the observer knew the distance passed over in three seconds, he would get the speed more accurately; yet more accurately, if he knew the distance passed over in one second; more accurately still, if he knew the distance passed over in half a second; and so on. The point to be noted and emphasised in this illustration is this: *the less* the time and the corresponding distance that can be observed, *the more nearly* will the observer obtain the actual speed of the train just at the moment when it is passing him.

3. To determine the speed of a falling body. Let a body fall vertically from rest. It is known that in t seconds from the time of starting, the body passes through $\frac{1}{2}gt^2$ feet. (Here g denotes a number whose approximate value is 32.2.) That is, if s denotes the number of feet through which the body falls in t seconds,

$$s = \tfrac{1}{2}gt^2.$$

As the body descends its speed is continually changing and growing greater; but at any particular instant it has *some* definite speed. Let it be required to find the speed after it has been falling for: (*a*) 4 seconds; (*b*) t_1 seconds.

(*a*) *To find speed after the body has been falling from rest for* 4 *seconds.* A method of getting an approximate value of this speed is as follows. Find the distance through which the body would fall in 4 seconds; then find the distance through which it would fall in a little more than 4 seconds. Therefrom deduce the average value of the speed from the end of the fourth second to the last instant (Note, Art. 1). This average speed may be taken as an approximate value of the speed at the end of the fourth second. The smaller the interval of time which is taken after the fourth second, the more nearly will the average speed for the interval be equal to the actual speed just at the end of the fourth second. This is also apparent from the following calculations:

Duration of fall, in seconds.	Length of fall, in feet.	Increase in time after 4 seconds. (In seconds.)	Corresponding increase in distance, in feet.	Average speed during increased time, in feet per second.	
4.	$8g$	—	—	—	—
4.1	$8.405g$	.1	$.405g$	$4.05g$ or	**130.41**
4.01	$8.04005g$	.01	$.04005g$	$4.005g$	**128.961**
4.001	$8.0040005g$	.001	$.0040005g$	$4.0005g$	**128.8161**
4.0001	$8.000400005g$	.0001	$.000400005g$	$4.00005g$	**128.80161**
. . .					
$4+h$	$(8+4h+\frac{1}{2}h^2)g$	h	$(4h+\frac{1}{2}h^2)g$	$\left(4+\frac{h}{2}\right)g$	$\mathbf{128.8+16.1\times h}$

It is evident that the less the increase given to the 4 seconds, the more nearly does the average speed during this additional time approach to 128.8 feet per second. The last line of the table shows that, no matter how short a time h may be, the average speed during this time has a definite value, namely $(128.8 + 16.1 \times h)$ feet per second. The number in brackets becomes more and more nearly equal to 128.8 when h is made smaller and smaller; the difference between it and 128.8 can be made as small as one pleases, merely by decreasing h, and will become still less when h is further diminished. Since the number $(128.8 + 16.1 \times h)$ *behaves in this way*, the speed of the falling body at the end of the fourth second is manifestly 128.8 feet per second.

(*b*) *To find the speed after the body has been falling for t_1 seconds.* Let s_1 denote the distance in feet through which the body has fallen in the t_1 seconds. It is known that

$$s_1 = \tfrac{1}{2} g t_1^2. \tag{1}$$

Let Δt_1 (read "delta t_1") denote any increment given to t_1, and Δs_1 denote the corresponding increment of s_1.

NOTE 1. Here Δt_1 does not mean $\Delta \times t_1$. *The symbol Δ is used with a quantity to denote any difference, change, or increment*, positive or negative (*i.e.* any increase or decrease), *in the quantity.* Thus Δx and Δy denote "increment of x," "increment of y," "difference in x," "difference in y."

Then
$$s_1 + \Delta s_1 = \tfrac{1}{2} g (t_1 + \Delta t_1)^2. \tag{2}$$

Hence, by (1) and (2),
$$\Delta s_1 = g t_1 \cdot \Delta t_1 + \tfrac{1}{2} g (\Delta t_1)^2.$$

$$\therefore \frac{\Delta s_1}{\Delta t_1} = g t_1 + \tfrac{1}{2} g \cdot \Delta t_1. \tag{3}$$

Here $\frac{\Delta s_1}{\Delta t_1}$ is the average speed for the time Δt_1 and the corresponding distance Δs_1. Now the smaller Δt_1 is taken, the more nearly will $\frac{\Delta s_1}{\Delta t_1}$ approximate to the actual speed which the falling body has at the end of the t_1th second. But when Δt_1 is taken smaller and smaller (in other words, when Δt_1 approaches nearer and nearer to zero), the second member of equation (3) approaches nearer and nearer to $g t_1$. Equation (3) also shows that $\frac{\Delta s_1}{\Delta t_1}$ can be made to differ as little as one pleases from $g t_1$, merely by taking Δt_1 small enough. Hence it is reasonable to conclude that at the end of the t_1th second

$$\text{the speed of the falling body} = g t_1 \text{ feet per second.} \tag{4}$$

Here t_1 may be any value of t. So it is usual to express conclusion (4) thus: the speed of a body that has been falling for t seconds is gt feet per second. This result (speed $= gt$ feet per second) is a general one, and can be applied to special cases. Thus at the end of the fourth second the speed is $g \times 4$ or 128.8 feet per second, as found in (*a*); at the end of 10 seconds the speed is $10\,g$ or 322 feet per second.

The two principal points to be noted in this illustration are:

(1) No matter what the value of Δt_1 may be, or how small Δt_1 may be, the quantity $\frac{\Delta s_1}{\Delta t_1}$ *has a definite value*, namely, $g t_1 + \frac{1}{2} g \cdot \Delta t_1$;

(2) When Δt_1 is taken smaller and smaller, $\frac{\Delta s_1}{\Delta t_1}$ gets nearer and nearer to $g t_1$; and the difference between them can be made as small as one pleases by giving Δt_1 a definite small value; this difference remains less than the assigned value when Δt_1 further decreases.

NOTE 2. The definite small value referred to in (2) can be easily found. For example, suppose that $\frac{\Delta s_1}{\Delta t_1}$ is to differ from gt_1 by not more than k say (k being any small quantity, as a millionth, or a million-millionth).

$$\text{Then} \quad \frac{\Delta s_1}{\Delta t_1} - gt_1 \overline{\gtrless} k. \quad \text{But } \frac{\Delta s_1}{\Delta t_1} - gt_1 = \tfrac{1}{2} g \cdot \Delta t_1 \text{ by (3).}$$

$$\therefore \tfrac{1}{2} g \cdot \Delta t_1 \overline{\gtrless} k; \text{ accordingly } \Delta t_1 \overline{\gtrless} \frac{2k}{g}.$$

NOTE 3. It should be observed, as shown by equation (3), that the value of $\frac{\Delta s_1}{\Delta t_1}$ depends upon the values of both t_1 and Δt_1. On the other hand, the value to which $\frac{\Delta s_1}{\Delta t_1}$ tends to become equal as Δt_1 decreases, depends (see (4)) upon t_1 alone. The quantity Δt_1 is any increment whatever of t_1, but it does not depend upon the value of t_1.

4. To determine the slope of the tangent *to the parabola* $y = x^2$*: (a) at the point whose abscissa is* 2; *(b) at the point whose abscissa is* x_1.

(*a*) Let VOQ, Fig. 1, be the parabola $y = x^2$, and P be the point whose abscissa is 2. Draw the secant PQ. If PQ turns about P until Q coincides with P, then PQ will take the position PT and become the tangent at P. The angle QPR will then become the angle TPR.

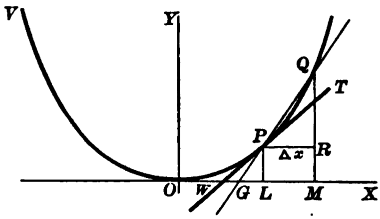

FIG. 1.

NOTE 1. This conception of a tangent to a curve has probably been already employed by the student in finding the equations of tangents to circles, parabolas, ellipses, and hyperbolas. The process generally followed in the analytic treatment of the conic sections is as follows: The equation of the secant PQ is found *subject to the condition* that P and Q are on the curve; then Q is supposed to move *along the curve* until it reaches P. The resulting form of the equation of the secant is the equation of the tangent at P. The calculus method (now to be shown) of finding tangents to curves is preferred by some teachers of analytic geometry; *e.g.* see A. L. Candy, *Analytic Geometry*, Chap. V.

Draw the ordinates PL and QM; draw PR parallel to OX. Let PR be denoted by Δx, and RQ by Δy. Then the slope of the secant PQ is $\frac{\Delta y}{\Delta x}$. $\left(\text{For } \tan RPQ = \frac{RQ}{PR}.\right)$

The following table shows the value of $\frac{\Delta y}{\Delta x}$ for various values of Δx.

Value of x.	Corresponding value of y.	Δx (Increase over x).	Δy (Increase over y).	Corresponding value of $\frac{\Delta y}{\Delta x}$.
2.	4.	—	—	—
2.1	4.41	.1	.41	**4.1**
2.01	4.0401	.01	.0401	**4.01**
2.001	4.004001	.001	.004001	**4.001**
2.0001	4.00040001	.0001	.00040001	**4.0001**
...	...	...	...	...
$2+h$	$4+4h+h^2$	h	$4h+h^2$	$4+h$

It is apparent from this table that the less Δx is, the more nearly does $\frac{\Delta y}{\Delta x}$ approach the value 4. The last line shows that, no matter how small Δx (or h) may be, $\frac{\Delta y}{\Delta x}$ has a definite value, namely $4+h$. This number becomes more and more nearly equal to 4 when h is made less and less; the difference between it and 4 can be made as small as one pleases, merely by decreasing h to a certain definite value, and will continue to be as small or smaller when h is further diminished. Because the number $4+h$ behaves in this way, it is evident that $\frac{\Delta y}{\Delta x}$ will reach the value 4 when Δx decreases to zero. Accordingly the slope of the tangent PT is 4; and hence angle TPR or PWL is $75° 57' 49''$.

(*b*) *To determine the slope of the tangent at the point whose abscissa is* x_1.

Let (Fig. 1) P be the point (x_1, y_1). Draw the secant PQ, and the ordinates PL and QM; draw PR parallel to OX. Let PR, the difference between the abscissas of P and Q, be denoted by Δx_1, and let RQ, the difference between the ordinates of P and Q, be denoted by Δy_1. Then

$$\text{tangent } QPR = \frac{RQ}{PR} = \frac{\Delta y_1}{\Delta x_1}.$$

If Q be moved along the curve toward P, the secant PQ will approach the position of PT, the tangent at P; at last, when Q reaches P, the secant PQ becomes the tangent PT. As Q approaches P, Δx_1 becomes less and less, and when Q reaches P, Δx_1 becomes zero. Conversely, as Δx_1 decreases, PQ approaches the position PT. Accordingly, the slope of the tangent PT can be determined by finding what the slope of the secant PQ, namely $\frac{\Delta y_1}{\Delta x_1}$, approaches when Δx_1 approaches zero.

$$y_1(=LP)=x_1^2,$$

$$y_1+\Delta y_1(=MQ)=(x_1+\Delta x_1)^2.$$

Hence, on subtraction,
$$\Delta y_1 = 2\,x_1\cdot\Delta x_1+(\Delta x_1)^2. \tag{1}$$

$$\therefore \frac{\Delta y_1}{\Delta x_1}=2\,x_1+\Delta x_1. \tag{2}$$

This equation shows that $\frac{\Delta y_1}{\Delta x_1}$ approaches nearer to $2\,x_1$ when Δx_1 decreases. It also shows that $\frac{\Delta y_1}{\Delta x_1}$ can be made to differ as little as one pleases from $2\,x_1$, merely by taking Δx_1 small enough, and that this difference will become smaller when Δx_1 is further diminished. (For instance, if it is desired that $\frac{\Delta y_1}{\Delta x_1}-2\,x_1$ be less than any positive small quantity, say e, it is only necessary to take Δx_1 less than e.) *Accordingly,*

$$\text{the slope of } PT \text{ (the tangent at } P) = 2\,x_1. \tag{3}$$

The two principal points to be noted in this illustration are:

(1) No matter what the value of Δx_1 may be, or how small Δx_1 may be, the quantity $\frac{\Delta y_1}{\Delta x_1}$ has a definite value, namely $2\,x_1+\Delta x_1$.

(2) When Δx_1 decreases, the quantity $\frac{\Delta y_1}{\Delta x_1}$ approaches the value $2x_1$; the difference between $\frac{\Delta y_1}{\Delta x_1}$ and $2\,x_1$ can be made as small as any number that may be assigned, by giving Δx_1 a definite small value; this difference remains less than the assigned value when Δx_1 further decreases.

NOTE 1. The value of $\frac{\Delta y_1}{\Delta x_1}$, as shown by Equation (2), depends upon the values of both x_1 and Δx_1. On the other hand, the value to which $\frac{\Delta y_1}{\Delta x_1}$ tends to become equal as Δx_1 decreases, depends (Equation (3)) upon x_1 alone. The value of Δx_1 does not depend upon the value of x_1; for Q (Fig. 1) may be taken anywhere on the curve.

NOTE 2. The method used in getting result (3) does not depend upon the particular value of x_1. The result is perfectly general, and may be expressed thus: "*the slope of the curve $y=x^2$ is $2\,x$.*" This general result can be used for finding the slope at particular points on the curve. For instance, if $x_1=2$, the slope is 4, as found in (a); if $x_1=-1$, the slope is -2, and accordingly, the angle made by the tangent with the x-axis is $116°\,34'$. (It is advisable to make a figure showing this.)

NOTE 3. In the infinitesimal calculus, as well as in other branches of mathematics, *it is very important for the student always to have a clear*

understanding of the meaning of the operations which he performs with numbers, and *to interpret rightly the numerical results obtained by these operations*. Thus, if it is stated that 6 men work 5 days at 2 dollars per day each, the numbers 6, 5, and 2 are treated by the operation called multiplication, and the number 60 is obtained. The calculator then applies, or interprets, this numerical result as meaning, not 60 men, or 60 days, but that the men have earned 60 dollars. In the curve above, $y = x^2$. This does not mean that at any point on the curve the ordinate is equal to the square on the abscissa, *i.e.* a length is equal to an area. By $y = x^2$ it is meant that the number of units of length in any ordinate is equal to the square of the number of units of length in the corresponding abscissa. Again, the result in Equation (3) does not mean that the slope of PT is twice OL. The result means that the number which is the value of the trigonometric tangent of the angle TPR is twice the number of units of length in OL.

Many persons who can perform operations of the calculus easily and accurately, cannot *correctly* or *confidently* interpret the results of these operations in concrete practical problems in geometry, physics, and engineering. Thus, some engineers who have had a fairly extended course in calculus discard it when possible, and solve practical problems by much longer and more laborious methods. Such a misfortune will not happen to those who early get into the habit of giving careful thought to finding out the real meaning of the operations and results of the calculus. They will not only "understand the theory," but they can use the calculus as a tool with ease and skill.

NOTE 4. In Fig. 1 let a point Q_1 be taken on the curve to the left of P, and draw the secant Q_1P. (The drawing for this note is left to the student.) It is obvious from the figure that the same tangent PT is obtained, whether the secant Q_1P revolves until Q_1 reaches P, or QP revolves until Q reaches P. This may also be deduced algebraically. Let the coördinates of Q_1 be $x_1 - \Delta x_1$, $y_1 - \Delta y_1$. [Here the Δx_1 and Δy_1 are not necessarily the same in amount as the Δx_1 and Δy_1 in (*b*).] Draw the ordinate Q_1M_1. Then

$$y_1 (= LP) = x_1^2,$$

$$y_1 - \Delta y_1 (= M_1Q_1) = (x_1 - \Delta x_1)^2.$$

Whence, it follows that $$\frac{\Delta y_1}{\Delta x_1} = 2x_1 - \Delta x_1.$$

Accordingly, when Δx_1 approaches zero, $\frac{\Delta y_1}{\Delta x_1}$ approaches the value $2x_1$.

NOTE 5. Thoughtful beginners in calculus are frequently, and not unnaturally, troubled by the consideration that when Δt_1 (Art. 3 *b*) is diminished to zero, $\frac{\Delta s_1}{\Delta t_1}$ has the form $\frac{0}{0}$; and likewise, when Δx_1 (Art. 4 *b*) becomes zero, $\frac{\Delta y_1}{\Delta x_1}$ becomes $\frac{0}{0}$. It is true that $\frac{0}{0}$ is indeterminate *in form;* and, if

it is presented *without any information being given* concerning the whence and the wherefore of its appearance, then its value cannot be determined. In the cases in Arts. 3, 4, however, there is given information which makes it possible to tell the meaning of the quantity $\frac{0}{0}$ that appears at the final stage of each of these problems. In these cases one knows how the quantities $\frac{\Delta s_1}{\Delta t_1}$ and $\frac{\Delta y_1}{\Delta x_1}$ *are behaving* when Δt_1 and Δx_1 respectively are approaching zero; and *by means of this knowledge* he can confidently and accurately state what these ratios *will become* when Δt_1 and Δx_1 actually reach zero.*

NOTE 6. Moreover, **it should be carefully noted** that *at the final stages* in the solution of the problems in Arts. 3 and 4, $\frac{\Delta s_1}{\Delta t_1}$ is *not* regarded as a fraction composed of *two* quantities, Δs_1 and Δt_1, but as a *single* quantity, namely the *speed* after t_1 seconds; likewise, that $\frac{\Delta y_1}{\Delta x_1}$ is then not regarded as a fraction at all, but as a single quantity, namely the *slope of the tangent* at P.

NOTE 7. The student should not be satisfied until he clearly perceives, and understands, that the *method* employed in solving the problems in Arts. 3 and 4 is *not a tentative one*, but is *general and sure*, and that the *results* obtained are *not indefinite or approximate*, but are *certain and exact.*

EXAMPLES.

1. Assuming the result in Art. 4 (*b*), namely, that the slope of the tangent at a point (x_1, y_1) on the curve $y = x^2$ is $2x_1$, find the slope and the angle made with the x-axis by the tangent at each of the points whose abscissas are

$$.5,\ 0,\ 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 4,\ -2,\ -3,\ -\tfrac{1}{2},\ -\tfrac{3}{4},\ -\tfrac{4}{3}.$$

2. In the curve in Ex. 1 find the coördinates of the points the tangents at which make angles of 20°, 30°, 45°, 60°, 85°, 115°, 145°, 160°, 170°, respectively, with the x-axis.

3. Draw figures of the following curves. Find the value of $\frac{\Delta y}{\Delta x}$ at any point (x, y) in the case of each curve; then find what $\frac{\Delta y}{\Delta x}$ is approaching when Δx approaches zero:

(*a*) $x^2 + y^2 = 16$; (*b*) $y = x^2 + x + 1$; (*c*) $y = x^3$;

(*d*) $y^2 = 8x$; (*e*) $9x^2 + 16y^2 = 144$; (*f*) $9x^2 - 16y^2 = 144$;

(*g*) $y^2 = 4px$; (*h*) $b^2x^2 + a^2y^2 = a^2b^2$; (*i*) $b^2x^2 - a^2y^2 = a^2b^2$.

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* The mathematical phraseology and notation employed to express these ideas is given in Chapter II.

[SUGGESTION. In (*a*), $(x + \Delta x)^2 + (y + \Delta y)^2 = 16$. It can then be deduced that $\frac{\Delta y}{\Delta x} = -\frac{2x + \Delta x}{2y + \Delta y}$.]

Compare the results found in (*g*), (*h*), and (*i*), with those found in analytic geometry.

4. Using the results obtained in Ex. 3, find the *slopes* and the *angles* made with the x-axis by the tangents in the following cases:

(*a*) The curve in Ex. 3 (*a*), at the points whose abscissas are

$$4,\ 2,\ 1,\ 0,\ -1.5,\ -3.5.$$

(*b*) The curve in Ex. 3 (*c*), at the points whose abscissas are

$$-3,\ -2,\ -1,\ 0,\ 1.5,\ 2.5.$$

(*c*) The curve in Ex. 3 (*d*), at the points whose abscissas are

$$0,\ 1,\ 2,\ 3,\ 6,\ 8.$$

(*d*) The curve in Ex. 3 (*e*), at the points whose abscissas are

$$0,\ 1,\ 2,\ 4,\ -.5,\ -1.5.$$

(*e*) The curve in Ex. 3 (*f*), at the points whose abscissas are

$$4,\ 8,\ 10,\ -5,\ -7.$$

5. Using the results obtained in Ex. 3, find the points on the curve in Ex. 3 (*a*) the tangents at which make angles 40° and 136° with the x-axis.

6. Do as in Ex. 5 for the curves whose equations are given in Ex. 3 (*c*), (*d*), (*e*), and (*f*).

7. Do some of the examples in Art. 59. Make careful drawings in each case.

5. To determine the area of a plane figure. A plane area, say *ABCD*, may be supposed to be divided into an infinitely great number of infinitely small rectangles. It will be seen later that the *limit* of the sum of these rectangles when they are taken smaller and smaller, is the area. The calculus furnishes a way to find this limit. Even at this stage in the study of the calculus the student can get some useful ideas concerning this problem by making a brief inspection of Art. 95, Exs. (*a*), (*b*), (*c*). [Art. 14 discusses the term "limit."]

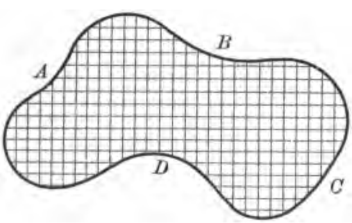

FIG. 2.

6. (*a*) **To find a function when its rate of change at any (every) moment is known,** or, in more general terms, when its *law of change* is known. In Art. 3 (*b*) a particular example has been given of this general problem, viz. to determine the rate of change of a function at any moment. The calculus not only provides a method of solving this general problem, but also provides a method of solving the inverse problem which is stated above.

(*b*) **To find the equation of a curve when its slope at any (every) point is known.** In Art. 4 (*b*) a particular example has been given of this general problem, viz. to determine the slope of a curve at any point on it. The calculus not only provides a method of solving this problem, but it also provides a method of solving the inverse problem which has just been stated. Problem (*b*) is a special case of problem (*a*), for the slope at a point on a curve really shows "the law of change" existing between the ordinate and the abscissa of the point (see Art. 26).

A brief inspection of Arts. 24–26, 97, 99, at this time, will repay the beginner.

NOTE. **Differential calculus and integral calculus.** The subject of infinitesimal calculus is frequently divided into two parts; namely, *differential calculus* and *integral calculus*. This division is merely a formal division; though oftentimes convenient, it is by no means necessary. Examples of the kind given in Arts. 2–4 formally belong to "the differential calculus," and those described in Arts. 5, 6, to "the integral calculus."

7. Elementary notions used in infinitesimal calculus. The problems used in Arts. 2–4 put in evidence some notions and methods, the consideration and development of which constitute an important part of infinitesimal calculus. These notions are:

(1) The notion of varying quantities which may approach as near to zero as one pleases, such as Δt_1 and Δx_1 in the last stages of the solution of the problems in Arts. 3 and 4.

(2) The notion of a varying quantity, such as $\frac{\Delta s_1}{\Delta t_1}$ in Art. 3 $\left(\text{or } \frac{\Delta y_1}{\Delta x_1} \text{ in Art. 4}\right)$, which approaches a fixed number when Δt_1 (or Δx_1) becomes more nearly equal to zero, and approaches in such a way that the difference between the varying quantity and the fixed number can be made to become, and remain, as small as one pleases, merely by decreasing Δt_1 (or Δx_1).

The infinitesimal calculus gives mathematical definiteness and exactness to these notions, and a convenient notation has been invented for dealing with them. From these notions, with the help of this notation, it has developed methods and obtained results which are of great service in such widely separated fields of study as geometry, astronomy, physics, mechanics, geology, chemistry, and political economy.

A review of certain notions of algebra is not only highly advantageous but absolutely necessary for a satisfactory understanding of the calculus and for good progress in its study. Accordingly, Chapter II. is devoted to the consideration of the notions of *a variable, a function, a limit,* and *continuity.*

NOTE. **Reference for collateral reading.** Perry, *Calculus for Engineers*, Preface, and Arts. 1–18.

CHAPTER II.

ALGEBRAIC NOTIONS WHICH ARE FREQUENTLY USED IN THE CALCULUS.

8. Variables. When in the course of an investigation a quantity can take different values, the quantity is called *a variable quantity*, or, briefly, *a variable*. For instance, in the example in Art. 3, the distance through which the body falls and its speed both vary from moment to moment, and, accordingly, are said to be variables. Again, if the x in the expression $x^2 + 3$ be allowed to take various values, then x is said to be a variable, and $x^2 + 3$ is likewise a variable. If a steamer is going from New York to Liverpool, its distance from either port is a variable.

NOTE 1. **Numbers and their graphical representation.** The measures of quantities are indicated by means of numbers. For instance, if a distance is 30 feet, its measure (when a foot is taken as the unit of measurement) is 30; and its measure is 360 when an inch is taken as the unit. When a quantity varies, the number which indicates its measure varies. Numbers which involve $\sqrt{-1}$ are *called* imaginary numbers; other numbers are said to be real numbers.* The (so-called) real numbers can be represented graphically on a straight line $L'OL$ extending to an infinite distance in both directions from O. Let unity be represented by some arbitrarily chosen

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* Real numbers are divided into two classes, *algebraic numbers* and *transcendental numbers*. Every (real) root of an algebraic equation, $ax^n + bx^{n-1} + \cdots + bx + m = 0$, with integral coefficients is called *an algebraic* (real) *number*. These numbers include integers, irrational numbers such as $\sqrt{2}$ and $\sqrt[5]{3}$, and fractional numbers formed from integers and irrational numbers. A real number which cannot be a root of an algebraic equation of the form described is called a *transcendental number*. A well-known number of this kind is π, the ratio of the circumference of a circle to its diameter. Transcendental numbers are irrational. There are far more transcendental numbers than algebraic. For an interesting brief elementary discussion on transcendental numbers see Klein, *Famous Problems in Elementary Geometry* (Beman and Smith's translation, Ginn & Co.), in particular, pages 51–54.

length, say OM. Let the distances of the points on the line be measured from O, and, according to the usual *convention*, let the distances of points on the *right* of O be regarded as positive (and be given a *plus* sign), and the distances of points on the *left* of O be regarded as negative (and be given a *minus* sign). To each point P on OL there corresponds a definite number,

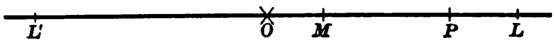

FIG. 3.

viz. the ratio $OP : OM$, the number n say; and to each number, for instance n, there corresponds a definite point P such that $OP = n \cdot OM$. *Positive* numbers are represented by the points on the *right* of O, and *negative* numbers by the points on the *left* of O. When a point moves along the line from O to L, it passes over every point from O to L in succession, and represents successively each number from zero to the ratio $OL : OM$. Some of these numbers are *integral*, such as 1, 3, 12; some are *fractional*, such as $\frac{1}{2}$, $\frac{4}{3}$, $\frac{11}{7}$; and some are *incommensurable*, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$, π. The value of an incommensurable number can be expressed by fractional numbers to as close an approximation to exactness as one pleases; and the corresponding point on $L'OL$ can be located as nearly to absolute corréctness as one pleases. For instance, $\sqrt{2} = 1.4142 \cdots$; accordingly, the point corresponding to $\sqrt{2}$ lies between the points corresponding to 1.4 and 1.5; between the points corresponding to 1.41 and 1.42; between the points corresponding to 1.414 and 1.415; and so on.

As to the graphical representation of imaginary numbers see Chrystal, *Algebra* (ed. 1886), Part I., Chap. XII., § 2.

In this course, with the exception of a few instances, only real numbers are met.

The value of a number without regard to sign is called its **absolute value.** Thus the absolute values of the numbers $1, -2, \frac{1}{2}, -\frac{1}{3}$ are $1, 2, \frac{1}{2}, \frac{1}{3}$. The absolute value of a number x is denoted by the symbol $|x|$.

NOTE 2. **Infinite numbers.** The student has a general idea of the set of numbers ordinarily called *finite numbers*. There is also a set of numbers each of whose (absolute) values is "greater than any number that can be named" or is "beyond all bounds." These numbers are said to be *infinitely great numbers* or *infinite numbers*. Finite numbers have each one distinct symbol at least, as 2, $\sqrt{2}$, $\frac{1}{3}$, $\frac{2}{3}$, or .4, etc.; but infinite numbers have each the *same* symbol, namely ∞, which is called "infinity."

Instead, however, of reading $x = \infty$, "x is equal to infinity," it is better to say "*x is infinitely great*," or "*x is infinite*," or "*x is beyond all bounds*." The phrase "is equal to infinity" may give the impression that ∞ denotes a single, definite, immense quantity; an impression which is erroneous. For instance, consider a number and its logarithm to base 10. Log $10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, $\log 1{,}000{,}000 = 6$, $\log 1{,}000{,}000{,}000{,}000 = 12$, and

so on. It is evident that when the logarithm is infinitely great, the corresponding number is also infinitely great. Now these infinitely great numbers are very different from each other; for when the logarithm becomes infinite, the corresponding number is much further along (so to say) in the set of infinite numbers. But both these numbers (the logarithm and the anti-logarithm) are then denoted by the same symbol, viz. ∞.*

NOTE 3. **References for collateral reading on numbers.** Echols, *Calculus*, Arts. 1–9; Harkness and Morley, *Introduction to the Theory of Analytic Functions*, Chaps. I., II.; Whittaker, *Modern Analysis*, Chap. I.

9. Functions. The area of a circle varies when the radius varies, and when the radius has a definite value, the area has a corresponding definite value. The volume of a cube varies with its length, and when the length has a definite value, the volume has a corresponding definite value. To the sine of an angle there correspond certain definite values of the angle. The number of deaths per year in a city depends, in some measure, upon the number of people in it, and in each city there is a definite number of deaths per year. These facts illustrate the following definition: *When two variables are so related that to a definite value of one of them there corresponds a definite value (or values) of the other, the second is said to be* **a function** *of the first.* The first is sometimes called *the argument* of the function.

The following definition of a function may also be used: When two variables are so related that the value of one of them depends upon the value of the other, the first is said to be *the dependent variable* or to be *a function of the second,* and the second is said to be *the independent variable* or to be *the argument of the function.*

The exact relation between the variables may, or may not, be capable of definite statement.

If
$$y = 2x^2 + 3x - 7, \tag{1}$$
the value of y varies when x varies, and the value of y depends upon the value of x; here y is the dependent variable (or the function), and x is the independent variable (or the argument). On the other hand, the value of x varies when y varies, and the value of x depends upon the value of y. If

* The two infinitely great numbers here referred to are compared in Appendix, Note C (Art. 3, Ex. 1).

y be allowed to vary, x *must* vary in a manner to suit; in such a case y is the independent variable, and x is the dependent variable (or the function). Precisely the same remarks may be made if x and y are connected by the relation

$$x^2y + y^2x + x^2 - 3y + 7 = 0. \tag{2}$$

When a relation connecting two variables is given, it does not matter, except in so far as convenience is concerned, which variable is regarded as independent. When one variable is chosen as the independent variable, the other must be considered the function.

The value of a variable may depend upon the values of two or more variables, or it may have a definite value (or several definite values) when two or more other variables have definite values. In such a case the first variable is said to be *a function of the other two*, or the first variable is said to be *a dependent variable*, and the other two are said to be *independent variables*.

Thus the distance which a vessel sails from a port depends both upon the time since departure and upon the speed; in other words, the distance sailed is a function of the time and the speed. Again, if

$$y^2 + 3z^2 + x^2 + 11 = 0,$$

z is a function of x and y.

NOTE. On the term "function," read Gibson, *Calculus*, § 11.

10. Constants. When a quantity remains unchanged during the course of an investigation, the quantity is said to be **a constant.** Thus in the case of a steamer going from New York to Liverpool the distance of the steamer from either port is variable, and the distance between the ports is constant. If a quantity has the same value in every investigation, it is said to be **an absolute constant;** if it has a particular value in one investigation, and another value in a second investigation, and so on, it is said to be **an arbitrary constant.**

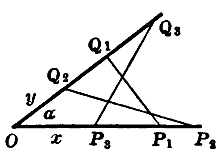

FIG. 4.

Thus g (Art. 3), the ratio π, 2, $\frac{1}{3}$ are absolute constants. In each of the triangles (Fig. 4) having a common vertical angle α, let x and y denote the lengths of the sides containing this angle, and let A denote the area of the triangle. Then, by trigonometry, $A = \frac{1}{2}xy \sin \alpha$. Here A, x, and y are variables, $\frac{1}{2}$ is an absolute constant, and α is an arbitrary constant.

11. Classification of Functions.

A. **Explicit and implicit functions.** When a function is expressed directly in terms of the dependent variable, like y in equation (1), Art. 9, the function is said to be *an explicit function.* When the function is not so expressed, as in equation (2), Art. 9, it is said *to be an implicit function.* If relation (2), Art. 9, were solved for y, then y would be expressed as an explicit function of x.

B. **Algebraic and transcendental functions.** Functions may also be classified according to the operations involved in the relation connecting a function and its dependent variable (or variables). When the relation involves only a finite number of terms, and the variables are affected only by the operations of addition, subtraction, multiplication, division, raising of powers, and extraction of roots, the function is said to be *algebraic;* in all other cases it is said to be *transcendental.* Thus $2x^2 + 3x - 7$, $\sqrt{x} + \frac{1}{x}$, are algebraic functions of x; $\sin x$, $\tan(x + a)$, $\cos^{-1} x$, l^x, e^{2x}, $\log x$, $\log 3x$, are transcendental functions of x. The elementary transcendental functions are the *trigonometric, anti-trigonometric, exponential,* and *logarithmic.* Examples of these have just been given.

C. **Continuous and discontinuous functions.** A discussion on this *exceedingly important* classification of functions is contained in Art. 16.

12. Notation. In general discussions variables are usually denoted by the last letters of the alphabet, $x, y, z, u, v, \cdots$, and constants by the first letters, $a, b, c, \cdots$.

The mere fact that a quantity is a function of a single variable, x, say, is indicated by writing the function in one of the forms $f(x)$, $F(x)$, $\phi(x)$, $\cdots$, $f_1(x)$, $f_2(x)$, $\cdots$. If one of these occurs alone, it is read "a function of x" or "some function of x"; if several are together, they are read "the f-function of x," "the F-function of x," "the phi-function of x," $\cdots$. The letter y is often used to denote a function of x.

The fact that a quantity is a function of several variables, $x, y, z, \cdots$, say, is indicated by denoting the quantity by means of some one of the symbols, $f(x, y)$, $\phi(x, y)$, $F(x, y, z)$, $\psi(x, y, z, u)$, $\cdots$. These are read "the f-function of x and y," "the phi-function of x and y," "the F-function of x, y, and z," etc.

Sometimes the exact relation between the function and the dependent variable (or variables) is stated; as, for example,

$f(x) = x^2 + 3x - 7$, or $y = x^2 + 3x - 7$; $F(x, y) = 2e^x + 7e^y + xy - 1$.

In such cases the f-function of any other number is obtained by substituting this number for x in $f(x)$, and the F-function of any two numbers is obtained by substituting them for x and y respectively in $F(x, y)$. Thus

$$f(z) = z^2 + 3z - 7,\ f(4) = 4^2 + 3 \cdot 4 - 7 = 21;$$

$$F(t, z) = 2e^t + 7e^z + tz - 1,\ F(2, 3) = 2e^2 + 7e^3 + 5.$$

EXAMPLES.

1. Calculate $f(2)$ and $f(.1)$ when $f(x) = 3\sqrt{x} + \frac{2}{x} + 7x^2 + 2$. Write $f(y)$, $f(m)$, $f(\sin x)$.

2. Calculate $f(2, 3)$, $f(-2, 1)$, and $f(-1, -1)$ when $f(x, y) = 3x^2 + 4xy + 7y^2 - 13x + 2y - 11$. Write $f(u, v)$, $f(\sin x, 2)$.

3. Calculate z as a function of x when $y = f(x) = \frac{2 + 3x}{4 - 7x}$ and $z = f(y)$.

4. Given that $f(x) = x^2 + 2$ and $F(x) = 4 + \sqrt{x}$, calculate $f[F(x)]$ and $F[f(x)]$.

5. If $f(x, y) = ax^2 + bxy + cy^2$, write $f(y, x)$, $f(x, x)$, and $f(y, y)$.

6. If $y = f(x) = \frac{ax + b}{cx - a}$, show that $x = f(y)$.

7. If $y = \phi(x) = \frac{2x - 1}{3x - 2}$, show that $x = \phi(y)$, and that $x = \phi^2(x)$, in which $\phi^2(x)$ is used to denote $\phi[\phi(x)]$.

8. If $f(x) = \frac{x + 1}{x - 1}$, show that $f^2(x) = x$, $f^3(x) = x$, $f^4(x) = x$, etc., in which $f^2(x)$ is used to denote $f[f(x)]$, $f^3(x)$ to denote $f\{f[f(x)]\}$, etc.

9. If $f(x) = \frac{x - 1}{x + 1}$, show that $\frac{f(x) - f(y)}{1 + f(x) \cdot f(y)} = \frac{x - y}{1 + xy}$.

NOTE. *Notation for inverse functions.* The student is already familiar with the trigonometric functions and their inverse functions, and with the notation employed; thus, $y = \tan x$, and $x = \tan^{-1} y$. In general if y is a function of x, say $y = f(x)$, then x is a function of y. The latter is often expressed thus: $x = f^{-1}(y)$. For instance, if $y = \log x$, $x = \log^{-1}(y)$. This notation was explained in England first by J. F. W. Herschell in 1813, and at an earlier date in Germany by an analyst named Burmann. See Herschell, *A Collection of Examples of the Application of the Calculus of Finite Differences* (Cambridge, 1820), page 5, Note.

13. Geometrical representation of functions of one variable.* The fact that the relation between a function and its independent variable can be made manifest to the eye by means of a curve, is familiar to students of analytic geometry. For instance, let $y=2x+3$, and draw a line MN having a slope 2 and an intercept 3 on the y-axis. The line MN may be regarded as a *picture* or *geometrical representation* of the function $2x+3$. For any value of x the length of the ordinate drawn from the corresponding point on the x-axis to the line MN will give the value of the function $2x+3$.

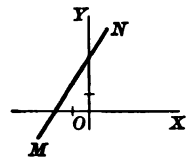

FIG. 5.

How to draw the curve whose equation is $y=f(x)$, or, *what is the same thing*, how to picture or represent the function $f(x)$, or make **a graph of the function $f(x)$**, is one of the fundamental problems in analytic geometry. For instance, if the function y and the independent variable x have the relation $x^2+y^2=25$, then the curve which represents the function y, *i.e.* $\sqrt{25-x^2}$, is the circle whose centre is at the origin of coördinates and whose radius is 5 units in length.

EXAMPLES.

1. Draw the curve which represents the function $\sqrt{x}$. (That is, make the graph of $\sqrt{x}$, or draw the curve whose equation is $y=\sqrt{x}$.)

2. Draw the curves which represent the following functions, and write the equations of the curves:

(*a*) $3x-5$, (*b*) $\frac{1}{2}x+3$, (*c*) $\sqrt{49-x^2}$, (*d*) $4x^2$,

(*e*) $\sqrt{x^2-49}$, (*f*) $\sin x$, (*g*) $\cos x$, (*h*) $\tan x$.

On the one hand, *some properties of the graph of a function can be predicted* by means of what is termed "a discussion of the equation" involving the function and its dependent variable. For instance, if $x^2+y^2=16$, it is obvious that for any value of x there are two values of y which are numerically equal but opposite

* **References for collateral reading and review** on this topic: Hall, *Introduction to Graphical Algebra;* Chrystal, *Introduction to Algebra*, Chaps. V., XXV.; Gibson, *Calculus*, Chaps. II., III.; Tanner and Allen, *Analytic Geometry*, Chap. III. and Art. 49; and other texts on algebra and on analytic geometry.

in sign; hence the graph of y must be symmetrical about the x-axis. It is also evident that y has real values when x is between -4 and $+4$, and that y is zero when x is -4 or $+4$, and that y is imaginary when x is less than -4 and greater than $+4$. Hence the graph of y must lie between the vertical lines drawn at $x = -4$ and $x = 4$.

On the other hand, *important properties of a function can be discovered by an inspection of its representative curve* or graph.* Thus, in Arts. 63, 64, 74, 76, etc., graphs are used in the investigation of functions; especially because these curves tend to make the investigations simpler and clearer for beginners. These investigations of functions can be conducted, however, without any reference to representative curves. But geometrical representation of functions serves to illustrate and emphasise properties already known about the functions, and also serves as a means for the discovery of new properties.

NOTE. The geometrical representative of a function of two variables is a surface. Thus, if $z = \sqrt{a^2 - x^2 - y^2}$, the geometrical representative of z is a sphere of radius a with its centre at the origin of coördinates. See Art. 79, Note 1.

14. Limits. The notion that varying quantities may have *fixed limiting values* is very important, and should be clearly understood before entering upon the study of the calculus.

EXAMPLES.

1. The number 0.3333 ⋯, *i.e.* the sum of the geometric series

$$\tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \cdots, \tag{1}$$

varies with n, the number of terms in the series. The greater n is, the more nearly does the sum of this series come to $\frac{1}{3}$. If n is a million, the sum of the series is very near to $\frac{1}{3}$; if n is a million million, the sum is still nearer to $\frac{1}{3}$. But the sum of the series, even if the number of terms be infinite (*i.e.* greater than any number that can be named), can never be actually $\frac{1}{3}$. Nevertheless, *if any positive number, say* ϵ, no matter how near to zero but not actually zero, *be assigned, it is possible to take a number of terms, n say,*

* It is not the case that every function of one variable can be represented by a curve. The question, what are the circumstances under which it is impossible for a graph of a function to be drawn, will not be discussed here. (See Harnack, *Calculus*, Art. 15.) All the functions which the student will meet in this course can be represented by curves.

of (1), such that *the difference* between the sum of these n terms and $\frac{1}{3}$ *will be less than the assigned number* ϵ, and *will remain less than* ϵ for any greater number of terms. (Thus, if ϵ be .000001, then n can be 6 or any greater number.) This is expressed mathematically by saying "the limiting value (or, briefly, 'the limit') of the sum of the series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$, as the number of terms approaches an infinitely great number, is $\frac{1}{3}$."

2. In Fig. 1, Art. 4, let Q move along the curve until it comes to P. When Q moves toward P, PR or Δx becomes smaller. If any length be assigned (say .00001 inch), then Q can move so near to P that Δx shall become and remain less than .00001 inch. Finally, when Q reaches P, Δx becomes zero. All this is expressed in mathematical language by saying "the limit of Δx, when Q approaches P, is zero." Similarly, the limit of the chord PQ, as the arc QP approaches zero, is zero.

3. In Fig. 1, Art. 4, when Q moves toward P the angle PGX approaches the angle PWX. If any angle be assigned, say ϵ'', Q can be made to approach so near to P that the difference between the angles PGX and PWX will be less than ϵ'', and will continue to be less than ϵ'' when Q approaches still nearer to P. Finally, when Q reaches P, PGX becomes PWX. This is expressed in mathematical terms by saying "the limit of the angle PGX, when Q approaches P, is PWX."

4. Show that in Fig. 1, Art. 4, "the limit of the angle TPQ, as the arc PQ approaches zero, is zero."

5. Let a regular polygon of n sides be inscribed in a circle. When the number of sides is increased the length of the polygon becomes more and more nearly equal to the length of the circle. These lengths can never become exactly equal; but the difference between them can be made less than any positive number that may be assigned, simply by increasing the number of the sides; and this difference will continue to remain less than the assigned number when the number of the sides is further increased. This is expressed mathematically thus: "The limit of the length of the perimeter of a regular polygon inscribed in a circle, as the number of sides approaches an infinitely great number, is the length of the circle."

6. Show that "the limit of the area of a regular polygon inscribed in a circle, as the number of sides approaches an infinite number, is the area of the circle."

7. Enunciate and prove propositions similar to Exs. 5, 6, about a circle and a circumscribing regular polygon.

8. The number $\frac{1}{(-2)^n}$ varies with n. As n increases this number decreases and approaches nearer and nearer to zero. It can never reach zero. But by increasing n the difference between the number and zero can be made less than any positive number that may be assigned; and on further increasing n this difference will continue to be less than the assigned number. Accordingly, the limit of the variable number $\frac{1}{(-2)^n}$, as n approaches an infinitely great value, is zero.

9. Show that the limit of $\frac{1}{2^n}$, as n approaches an infinite number, is zero.

(The number in Ex. 8 is alternately positive and negative according as n is even or odd; hence, it is alternately greater and less than its limit. The number in Ex. 9 is always positive, and, accordingly, is always greater than its limit.)

10. Show that the limit of the sum $2 - 1 + \frac{1}{2} - \cdots$ to n, terms, as n increases beyond all bounds, is $\frac{4}{3}$.

11. In Ex. (a), Art. 4, $\frac{\Delta y}{\Delta x}$ varies with Δx, and approaches 4 as Δx approaches zero. By decreasing Δx the difference between $\frac{\Delta y}{\Delta x}$ and 4 can be made less than any positive number that may be assigned, and will remain less than this number when Δx continues to decrease. That is, the limit of $\frac{\Delta y}{\Delta x}$, as Δx approaches zero, is 4.

Show that in Ex. (b), Art. 4, the limit of $\frac{\Delta y}{\Delta x}$, as Δx approaches zero, is $2x$.

NOTE 1. In each of these cases $\frac{\Delta y}{\Delta x}$ finally reaches its limit. In Ex. 10 the variable sum can never reach its limit.

12. In Ex. (b), Art. 3, $\frac{\Delta s}{\Delta t}$ varies with Δt, and approaches gt as Δt approaches zero. By decreasing Δt the difference between $\frac{\Delta s}{\Delta t}$ and gt can be made less than any positive number that may be assigned, and will remain less than this number when Δt continues to decrease. Accordingly, the limit of $\frac{\Delta s}{\Delta t}$, as Δt approaches zero, is gt.

In Ex. (a), Art. 3, the limit of $\frac{\Delta s}{\Delta t}$, as Δt approaches zero, is 128.8.

In each of these cases $\frac{\Delta s}{\Delta t}$ can reach its limit.

13. (a) Show that the limit of $\frac{\sin\theta}{\theta}$, as θ approaches zero, is 1.

(b) Show that the limit of $\frac{\tan\theta}{\theta}$, as θ approaches zero, is 1.

(c) Show that the limit of $\cos\theta$, as θ approaches zero, is 1.

(d) Show that the limit of $\sin\theta$, as θ approaches zero, is 0.

(e) Show that the limit of $\sin\theta$, as θ approaches $\frac{\pi}{2}$, is 1.

14. Show that the limit of $\frac{x^2 - a^2}{x - a}$, when x approaches a, is $2a$.

NOTE 2. The limit of a constant is the constant itself.

Definition of a limit. *Let there be a function of a variable, and let the variable approach a particular value. If, at the same time as the variable approaches the particular value, the function also approaches a fixed constant in such a way that the absolute value of the difference between the function and the constant may be made less than any positive number that may be assigned; and if, moreover, this difference continues to remain less than the assigned number when the variable approaches still nearer to the particular value chosen for it; then* **the constant is the limit of the function when the variable approaches the particular value.**

NOTE 3. *This definition may be expressed in a slightly different form*, viz.:

Let the variable x approach a particular value a; **if** (1) as x approaches nearer to $a, f(x)$ approaches nearer to a fixed constant A, and **if** (2) a number, say ϵ, being taken as small as one pleases, zero excepted, it is possible to find a number h such that $|f(a+h) - A| < |\epsilon|$, and **if** (3) as h decreases, the quantity $|f(a+h) - A|$ continues to be less than ϵ; **then** *A is said to be a limit of $f(x)$ as x approaches a.*

NOTE 4. If the difference between the varying function and the fixed constant can actually become zero, the limit is *an attainable limit;* if the difference can never be zero, the limit is *an unattainable limit.*

Ex. Mention the variables and functions in Exs. 1–14 above, which have attainable limits.

NOTE 5. A variable or a function may be always less than its limit, or always greater than its limit, or sometimes greater and sometimes less than its limit.

Ex. Examine the variables and functions in Exs. 1–14 with respect to this matter.

NOTE 6. The importance of the phrase "*and continues to remain less*" in the definition of a limit should be clearly apprehended. A variable function may approach a constant value by a series of advances and retreats. Thus a point may move on the line OX from A to M in the following way.

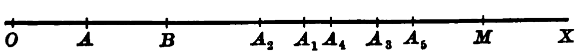

FIG. 6.

It may go forward from A to A_1, then back to A_2, then forward to A_3, then back to A_4, then forward to A_5, and so on. While on the whole it is getting nearer to M, still its distance from M does not continue to remain less than an assigned value. For instance, after it arrives at A_3 it goes back to A_4.

The idea of a limit, which has already been applied practically by the student in arithmetic, algebra, geometry, and trigonometry, plays a very great and important part in calculus.

15. Notation. The limit of a variable quantity, and the condition under which this limit is approached, are expressed by means of a certain mathematical shorthand. Thus the last sentence in Ex. 1, Art. 14, is expressed:

$$\text{Lim}_{n \doteq \infty} \left(\tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \cdots\right) = \tfrac{1}{3}.$$

The result found in Ex. 11 is expressed:

$$\text{Lim}_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 4.$$

The result found in Ex. 13 (*e*) is expressed:

$$\text{Lim}_{\theta \doteq \frac{\pi}{2}} \sin \theta = 1.$$

The symbol $\doteq$ is placed between a variable and a constant in order to indicate that the variable approaches the constant as a limit. Thus $\theta \doteq \frac{\pi}{2}$ above, means that θ approaches $\frac{\pi}{2}$ as a limit.

NOTE. The symbol $\doteq$ is used to indicate *an approach to equality.* The symbol $=$ is used by many instead of $\doteq$ to indicate the same idea. Various other notations are also employed.*

Ex. Express the results in Exs. 1–14 in the mathematical manner of writing.

16. Continuous variables. Continuous functions. Discontinuous functions. *A number x is said* **to vary continuously** *from the value a to the value b when in changing from a to b it takes each value between a and b once and only once.*

Thus (Fig. 6) let $OA = a$, $OB = b$, and x denote the distance from O of any point on OX. If a point moves along OX from A to B, then x varies continuously from a to b, and is said to be *a continuous variable.* The distance through which a body falls, varies continuously from when the body starts

* Professor Echols of the University of Virginia advocates the use of the symbol £ for the term "limit" and the symbol $(=)$ in preference to $\doteq$. See his *Calculus* (Holt & Co.), preface and Arts. 12, 13.

until it stops, and accordingly is a continuous variable. Again, if a point move along a circle, both the arc and the chord measured from a fixed point on the circle to the moving point vary continuously, *i.e.* are continuous variables. In the case of a steamer going straight ahead from New York to Liverpool without stopping its distance from New York is a continuous variable; so also is its distance from Liverpool.

A function $f(x)$ is said to be **a continuous function** *of x for all values of x from $x = a$ to $x = b$, when it satisfies the following conditions:*

(1) Its absolute value does not become infinite for any value of x between a and b;

(2) The corresponding change in $f(x)$ is also infinitely small when an infinitely small change is made in x while the value of x lies between a and b. In other words, if the value of $f(x)$ does not take a sudden jump of either a finite or infinite amount when x changes by only an infinitely small amount at any value between a and b.

NOTE 1. The second condition is expressed more formally and rigorously in Note 7. The notion of a continuous function becomes clearer, and is developed more fully, by means of investigations in the calculus.

Ex. 1. Let $f(x) = x^2 + 3x - 7$.

This function does not become infinite for any finite value of x; accordingly, condition (1) is satisfied. Let x_1 be any finite value of x, and h be an infinitely small change which is made in x when $x = x_1$. Then

$$f(x_1) = x_1^2 + 3x_1 - 7, \text{ and } f(x_1 + h) = (x_1 + h)^2 + 3(x_1 + h) - 7.$$

$$\therefore f(x_1 + h) - f(x_1) = h(2x_1 + h + 3).$$

The first member is the change in the function corresponding to the infinitely small change h in x when $x = x_1$. The second member can evidently be made as near zero as one pleases simply by decreasing h, and thus is infinitely small; accordingly $f(x)$ satisfies condition (2). Thus $f(x)$ satisfies both conditions (1) and (2), and, accordingly, is a continuous function of x for all finite values of x.

If, in the case of a function $f(x)$, either of the conditions (1) and (2) is not fulfilled when x has a particular value, say $x = c$, then the function $f(x)$ is said to be **discontinuous** *for the value $x = c$, or, more briefly, discontinuous at c.*

Ex. 2. Let $f(x) = \tan x$. (Here x denotes the number of radians in the angle.)

This function is continuous from $x = 0^c$ to $x = \frac{\pi}{3}$, and is continuous in many other intervals. But $\tan\frac{\pi}{2}$ is infinite, and accordingly, $\tan x$ is discontinuous for $x = \frac{\pi}{2}$.

The change that $\tan x$ makes when x passes through the value $\frac{\pi}{2}$, should also be noted. Let h denote a quantity exceedingly near to zero.

Then $\tan\left(\frac{\pi}{2} - h\right) =$ an exceedingly great *positive* quantity,

and $\tan\left(\frac{\pi}{2} + h\right) =$ an exceedingly great *negative* quantity.

$\therefore \tan\left(\frac{\pi}{2} - h\right) - \tan\left(\frac{\pi}{2} + h\right) =$ the sum of two exceedingly great positive quantities.

Thus, when x makes an exceedingly small change in passing from one side of the value $\frac{\pi}{2}$ to the other, the function $\tan x$ makes a tremendous jump and changes by an exceedingly great amount. This is also apparent from an inspection of the representative curve of $y = \tan x$.

NOTE 2. When $x = \frac{\pi}{2}$, $\tan x$ may be *either* an infinitely great positive quantity *or* an infinitely great negative quantity. If x is increasing from 0^c to $\frac{\pi}{2}$, then, when x reaches the value $\frac{\pi}{2}$, $\tan x$ has an infinitely great *positive* value; but if x is decreasing from π to $\frac{\pi}{2}$, then, when x reaches the value $\frac{\pi}{2}$, $\tan x$ has an infinitely great *negative* value. Thus *the value of* $\tan x$ for $x = \frac{\pi}{2}$, *depends upon the way in which* x *has approached to the value* $\frac{\pi}{2}$.

NOTE 3. An instance of a function which takes a sudden *finite* jump is given in Ex. 3.

Note 4. In a first course in calculus the student will meet, in general, with functions only when they are continuous. The remainder of this article is not absolutely necessary for simple work in calculus; but it may interest a beginner, and will show him the necessity there is for distinguishing functions as continuous and discontinuous.

Ex. 3. Examine the function $f(x) = 2(4^{\frac{1}{x-3}} - 1) \div (4^{\frac{1}{x-3}} + 1)$ when $x = 3$. Let h be a quantity as near zero as one pleases.

$$f(3-h) = 2\,\frac{4^{\left(-\frac{1}{h}\right)} - 1}{4^{\left(-\frac{1}{h}\right)} + 1} = 2\,\frac{\frac{1}{4^{\frac{1}{h}}} - 1}{\frac{1}{4^{\frac{1}{h}}} + 1};\quad f(3+h) = 2\,\frac{4^{\frac{1}{h}} - 1}{4^{\frac{1}{h}} + 1} = 2\,\frac{1 - \frac{1}{4^{\frac{1}{h}}}}{1 + \frac{1}{4^{\frac{1}{h}}}}.$$

When h approaches zero, $\frac{1}{h}$ approaches an infinite value; then $4^{\frac{1}{h}}$ approaches an infinite value, and $\frac{1}{4^{\frac{1}{h}}}$ approaches zero. Accordingly, when h is very nearly zero

$$f(3-h) = -2 \text{ nearly, and } f(3+h) = +2 \text{ nearly.}$$

Therefore, when h is very small, $f(3+h) - f(3-h)$ is exceedingly near to 4. Accordingly, when x in changing passes through the value 3, $f(x)$ changes by the amount 4; and thus $f(x)$ is discontinuous for $x = 3$.

NOTE 5. When (Ex. 3) $x = 3$, $f(x)$ may be either $+2$ or -2. If x is increasing from 0 toward 3, then, when $x = 3$, $f(x) = -2$; but if x is decreasing from 4 toward 3, then, when $x = 3$, $f(x) = +2$. Thus the value of $f(x)$ for $x = 3$ *depends upon the way in which x has approached the value* 3.

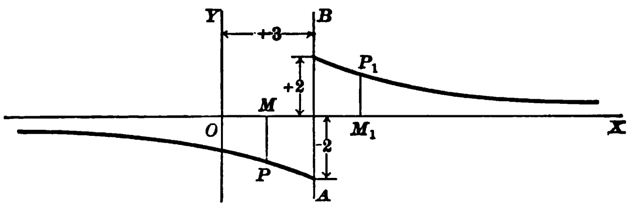

FIG. 7.

The representative curve is shown in Fig. 7. When x is moving toward the right the ordinate PM of the curve arrives at the line AB (the line $x = 3$), with the value -2, and leaves this line with the value $+2$. When x is moving toward the left the ordinate P_1M_1 of the curve arrives at AB with the value $+2$, and leaves this line with the value -2.

NOTE 6. *A simple instance of a discontinuous function in nature. The temperature* of a body (whether solid, liquid, or gaseous) subjected to heat depends upon *the quantity of heat* which has been received (or absorbed) by the body. When heat is added to a solid body the temperature of the body rises. If heat is added continuously, for a time *the temperature continues to rise.* But whenever the body begins to take the liquid form ("to melt"), *the temperature becomes stationary*, and it remains stationary (while heat is being added) until the body is completely liquefied. Then the temperature begins to rise again, and continues to rise until the liquid begins to vaporize, when it again becomes stationary. The temperature remains stationary (while heat is being added) until the liquid has all been vaporized, when it again begins to rise, and continues to rise so long as the gas continues to receive heat. Thus in the case of a mass of matter subjected to heat the temperature of the matter, in general, is a continuous function of the heat which is absorbed. But there are two "breaks" in the continuity; these occur

when the matter is melting and when it is changing into the gaseous form. The matter arrives at the "melting point," or temperature of fusion, possessed of a certain amount of heat, and leaves that temperature possessed of an additional amount of heat;* it arrives at the "boiling point," or temperature of vaporisation, possessed of a certain amount of heat, and leaves that temperature possessed of an additional amount of heat.†

Ex. Make a sketch to illustrate this case. Lay off on a horizontal axis the successive temperatures of the body, and on a vertical axis the amounts of heat received by the body.

Ex. 4. Show that y is a continuous function of x when $x^2 + y^2 = 16$.

Ex. 5. Show that the function $\frac{1}{x}$ is discontinuous only when x is zero. Find the change in the function when x passes through zero from a negative value to a positive value, and find the change when x passes through zero from a positive to a negative value. The curve $y = \frac{1}{x}$ is an hyperbola whose branches are in the first and third quadrants, and whose asymptotes are the axes of coördinates.

Ex. 6. Given that $\frac{y-1}{y-2} = 5^{\frac{1}{x-1}}$, show that the function y is discontinuous for $x = 1$. Show that, for x increasing, when x reaches the value 1, y has the value 1, and when x leaves the value 1, y has the value 2.

Ex. 7. Examine the function y at its point of discontinuity when

$$y = a \cdot \frac{e^{\frac{1}{x-a}} - 1}{e^{\frac{1}{x-a}} + 1}, \text{ in which } e > 1.$$

Note 7. More formal (and more rigorous) definitions of continuous functions are the following:

A. A function $f(x)$ is said to be continuous throughout the interval from $x = a$ to $x = b$, when

(1) $f(x)$ does not become infinite for any value of x between a and b; and

(2) at any point in this interval, as x_1, it is always possible to find a value of h for which $|f(x_1 + h) - f(x_1)|$ is less than any number as small as one pleases, say ϵ, that may be assigned.

B. A function $f(x)$ is said to be continuous when $x = a$, if

(1) $f(a)$ is not infinite and has a definite value (or definite values).

(2) $\text{limit}_{x \doteq a} f(x) = f(a)$.

An inspection of the definition of a limit, Note 3, Art. 14, and a comparison of definitions *A* and *B*, show that conditions (2) are practically identical.

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* "Latent heat of fusion." (See text-books on Physics.)

† "Latent heat of vaporisation."

NOTE 8. The following important proposition can be deduced from the definitions of continuity in Note 7, viz.: If a function $f(x)$ is everywhere continuous in the interval from $x = a$ to $x = b$, and x_1, x_2 are two points in this interval, then as x goes through the range of values from x_1 to x_2 the function assumes, once at least, each value which lies between $f(x_1)$ and $f(x_2)$. In other words, the continuous function does not overleap any values intermediate between two values which it assumes. The meaning of this proposition will be made clearer by a reference to Figs. 5, 20 *a*, *b*, *c*. In Fig. 5, when x varies from 2 to 3, y takes all values from 7 to 9.

NOTE 9. **References for collateral reading.** On *Limits* and *Continuous and Discontinuous Functions:* Chrystal, *Algebra* (Ed. 1889), Part I., Chap. XV., Part II., Chap. XXV. (in particular §§ 1–13, 24, 26); Harkness and Morley, *Introduction to the Theory of Analytic Functions*, Chap. VI. (VII.); Lamb, *Calculus*, Chap. I.; Gibson, *Calculus*, Chaps. IV., V.; Harnack, *Calculus* (Cathcart's translation), §§ 9–19; Echols, *Calculus*, Arts. 12–25. Also see references for collateral reading, Art. 21. With reference to the topic in Note 8, see Whittaker, *Modern Analysis*, Art. 30.

CHAPTER III.

INFINITESIMALS, DERIVATIVES, DIFFERENTIALS, ANTI-DERIVATIVES, AND ANTI-DIFFERENTIALS.

17. In this chapter some of the principal terms used in the calculus are defined and discussed, and one of the main problems of the calculus is described. In the first study of the calculus it is better, perhaps, not to read all this chapter very closely, but after a cursory reading of it to proceed to Chapter IV., and, while working the examples in that chapter, to re-read carefully the articles of this chapter. These articles can also be reviewed most profitably when the special problems to which they are applied are taken up. **Articles 22, 23**, however, should be carefully studied before Chapter IV. is begun.

18. Infinitesimals, infinite numbers, finite numbers. *An infinitesimal is a variable which has zero for its limit.* (See definition of a limit, Art. 14.) That is, if α denote an infinitesimal,

$$\alpha \doteq 0, \quad \text{or} \quad \text{limit } \alpha = 0.$$

For instance, in Ex. (*a*), Art. 4, when PR is approaching zero it is an infinitesimal. So also, at the same time, are angle QPT and the triangle PQR. Again, when angle θ is an infinitesimal $\sin\theta$ and $\tan\theta$ are infinitesimal; $\cos\theta$ is an infinitesimal when θ is approaching $\frac{\pi}{2}$; when n is increasing beyond all bounds $1 \div 2^n$ is an infinitesimal.

NOTE. The infinitesimal of the calculus is not the same as the infinitesimal of ordinary speech. The latter is popularly defined as "an exceedingly small quantity," and is usually understood to have a fixed value. The infinitesimal of the calculus, on the other hand, is a variable which approaches zero in a particular way.

The following statements are in accordance with, or follow directly from, the definitions of a limit and an infinitesimal.

(1) The difference between a variable and its limit is an infinitesimal. That is, on denoting the variable by x and the limit by a,

if $\quad \text{limit } x = a, \quad$ *i.e.* if $\quad x \doteq a,$

then $\quad x = a + \alpha, \quad$ in which $\quad \alpha \doteq 0.$

(2) If the difference between a constant and a variable is an infinitesimal, then the constant is the limit of the variable. In symbols, if

$$x = a + \alpha,$$

in which $\quad \alpha \doteq 0,$

then $\quad x \doteq a,$

i.e. $\quad \text{limit } x = a.$

This principle has been employed in the exercises in Arts. 3, 4.

It is evident that the reciprocal of an infinitesimal approaches a number which is greater than any number that can be named, namely, an infinite number. Accordingly, **an infinite number** may be defined as *the reciprocal of an infinitesimal.* Numbers which are neither infinitesimal nor infinite are called **finite numbers.**

19. Orders of magnitude. Orders of infinitesimals. Orders of infinites. Let m and n each denote a number which may be finite, infinite, or infinitesimal. When the limiting value of the ratio $\frac{m}{n}$ is a finite number, m and n are said to be *finite with respect to each other* and to be of *the same order of magnitude;* when the limit of the ratio $\frac{m}{n}$ is either zero or infinity, m and n are said to be of *different orders of magnitude.*

For instance, 1,897,000,000 and .000001 are of the same order of magnitude. Tan 90° and tan 45° are of different orders of magnitude. Log x and x are of different orders of magnitude when x is an infinite number. (See Appendix, Note C, Art. 3, Ex. 1.)

That infinitesimals may be of different orders of magnitude is shown by the following illustration.

Suppose that the edge BL of the cube in Fig. 8 is divided into any number of parts, and that each part, as Bb, becomes infinitesimal. Through each point of division, as b, let planes be passed at right angles to BL. The cube is thereby divided into an infinite number of infinitesimal slices like Bd. Now suppose that the edge BA is divided like BL into parts like Bf which become infinitesimal, and let a plane be passed through each point of division f at right angles to BA. The slice Bd is thereby divided into an infinite number of infinitesimal parallelopipeds like Ck. Finally suppose that the edge BC is divided into parts which become infinitesimal like Bg, and that through each point of division, as g, a plane is passed at right angles to BC. Then Ck is thereby divided into an infinite number of infinitesimal parallelopipeds like kg. Since the limiting value of each of the ratios $\frac{DL}{Bd}$, $\frac{Bd}{Ck}$, $\frac{Ck}{kg}$, is infinite, the parallelopipeds DL, Bd, Ck, kg, are all of different orders of magnitude.

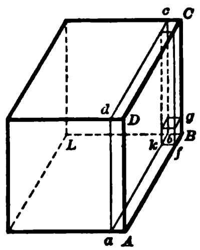

FIG. 8.

Definition. If α is an infinitesimal, and β is such that the limit of the ratio $\frac{\beta}{\alpha}$ is a finite number, then β is said to be *an infinitesimal of the same order* of magnitude as α, and β is said to be finite with respect to α. If β is such that the limit of the ratio $\frac{\beta}{\alpha^n}$, in which n is a positive integer, is finite, then β is said to be *an infinitesimal of the nth order* with respect to α.

In order to determine the orders of infinitesimals, it is necessary to take some one infinitesimal as a *standard* infinitesimal, and this standard infinitesimal is said to be of the first order. If the standard infinitesimal be denoted by α, then α^2, α^3, $\cdots$, α^n, are said to be infinitesimals of the second, third, $\cdots$, nth orders, respectively.

Infinite numbers, being reciprocals of infinitesimals, also have different orders of magnitude. With reference to the standard infinitesimal α, $\frac{1}{\alpha}$ is an infinitely great number. The numbers $\frac{1}{\alpha}$, $\frac{1}{\alpha^2}$, $\frac{1}{\alpha^3}$, $\cdots$, $\frac{1}{\alpha^n}$ (*i.e.* α^{-1}, α^{-2}, $\cdots$, α^{-n}), are said to be infinites of the first, second, $\cdots$, nth orders, respectively. If a number β be such that the limiting value of the ratio $\beta \div \frac{1}{\alpha}$ (*i.e.* $\beta\alpha$) is a

finite number, then β is said to be *an infinite of the first order,* and β and α^{-1} are said to be finite with respect to each other. If the limit of $\beta \div \frac{1}{\alpha^n}$ (*i.e.* $\beta\alpha^n$) is finite, then β is said to be *an infinite of order n.*

Theorems on infinitesimals. (*a*) The product of an infinitesimal α, and any finite number κ, namely $\kappa\alpha$, is an infinitesimal of the same order as α. This follows at once from the definition above.*

COR. 1. The sum of a finite number of infinitesimals of the same order is an infinitesimal of that order.

COR. 2. The algebraic sum of a finite number of infinitesimals is an infinitesimal.†

(*b*) The product of two infinitesimals, β and γ say, of orders m and n respectively, is an infinitesimal of order $m+n$. For, if α denote the standard infinitesimal, $\beta = \kappa_1\alpha^m$, $\gamma = \kappa_2\alpha^n$, and hence $\beta\gamma = \kappa_1\kappa_2\alpha^{m+n}$, which is an infinitesimal of order $m+n$. (Here κ_1 and κ_2 are finite numbers.)

(*c*) The quotient $\beta \div \gamma$ (see (*b*)) is an infinitesimal of order $m-n$.

N.B. These theorems are true for numbers of any magnitude, for finite and infinite numbers as well as for infinitesimals. The student can make the proofs.

EXAMPLES.

1. Let (Fig. 8) $AB = l$, and let Bg, Bf, fk, be infinitesimals of the first order. (i) Show that the volumes of Bd, Ck, and kg are infinitesimals of the first, second, and third orders respectively, with respect to the volume of DL. (ii) Show that, with respect to Ck, DL and Bd are infinites of the second and first orders respectively, and kg is an infinitesimal of the first order. (iii) Show that, with respect to kg, the volumes of Ck, Bd, and DL are infinites of the first, second, and third orders respectively.

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* The product of an infinitesimal and an *infinite* number may be infinitesimal, finite, or infinite, according to circumstances. Particular instances are given in Appendix, Note C.

† The limiting value of the sum of an *infinite* number of infinitesimals may be infinitesimal, finite, or infinite, according to circumstances. For simple illustrations see McMahon and Snyder, *Diff. Cal.*, page 12. Many instances in which this limiting value is finite will be found later in this book.

2. Show that, if Δx in Fig. 1 be an infinitesimal of the first order, then Δy is an infinitesimal of the first order, and the area of triangle PQR is an infinitesimal of the second order.

3. Show that if angle θ be an infinitesimal, $\sin\theta$ and $\tan\theta$ are infinitesimals of the same order as θ. (See Ex. 13, Art. 14, and Plane Trigonometry, Art. 83.) This is a very **important case** in infinitesimals.

4. Let θ denote one of the angles of a right-angled triangle, x the adjacent side, y the opposite side, and r the hypothenuse. Show that if θ is an infinitesimal of the first order, r and x are both finite, or both infinitesimals, or infinites of the same order; and show that if r is also an infinitesimal, y is an infinitesimal of an order one higher; and if r is an infinite, y is an infinite of an order one less; and if r is finite, y is an infinitesimal of the first order.

5. In the triangle in Ex. 4, in which θ is an infinitesimal of the first order, show that if r be an infinitesimal of order n, $r - x$ is of order $n + 2$.

$$\left[\text{SUGGESTION}: r^2 - x^2 = y^2 = r^2\sin^2\theta\,;\ \text{whence } r - x = \frac{r^2\sin^2\theta}{r+x}.\right]$$

6. In a circle of finite radius *the difference between the length of an infinitesimal arc of the first order and its chord is an infinitesimal of at least the third order.*

Let AB be an arc of a circle of finite radius r and centre O. Draw the chord AB and the tangents at A and B. These tangents meet at T; OT bisects arc AB, the chord AB, and the angle AOB. Let angle $AOC = \theta$; take θ for the standard infinitesimal. Then

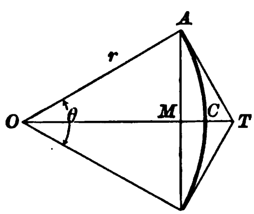

FIG. 9.

arc $AC = r\theta$ (trigonometry), an infinitesimal of first order, and

$AM = r\sin\theta$, an infinitesimal of first order (Ex. 3); also

$AT = r\tan\theta$, an infinitesimal of first order.

Now angle $MAT = \theta$. Hence, by Ex. 5, $AT - AM$ is an infinitesimal of the third order. But (arc $AC - AM$) $<$ $(AT - AM)$. Hence, 2(arc $AC - AM$), *i.e.* arc AB − chord AB, is an infinitesimal of at least the third order.

NOTE. The theorem stated in Ex. 6 holds for *any curve of finite curvature.* (See McMahon and Snyder, *Diff. Cal.*, Th. 4, page 27; also see Byerly, *Diff. Cal.*, Art. 165.)

20. Theorems on limits and infinitesimals. (*a*) *If two variables, x and y, be always equal, and if one of them, say x, approaches a limit a, then the other approaches the same limit;* that is,

if $\qquad x = y$, and $x \doteq a$, then $y \doteq a$.

For $\qquad x = a + \alpha$, in which $\alpha \doteq 0$;

hence $\qquad y = a + \alpha$; that is $y \doteq a$.

Ex. 1. The two members of Equation (3), Art. 3 (*b*), are always equal, and the second member approaches a limit gt_1 when Δt_1 approaches zero; hence the first member approaches the same limit.

Ex. 2. The two members of Equation (2), Art. 4 (*b*), are always equal, and the second member approaches a limit $2x_1$ when Δx_1 approaches zero; hence the first member approaches the same limit.

(*b*) *The limit of the sum of a finite number of variables, $x, y, z, \cdots$, is equal to the sum of their limits.*

For, if $\qquad x \doteq a,\ y \doteq b,\ \cdots,$

then $\qquad x = a + \alpha,\ y = b + \beta,\ \cdots,$

in which $\qquad \alpha \doteq 0,\ \beta \doteq 0\ \cdots.$

Hence $\qquad x + y + \cdots = a + b + \cdots + (\alpha + \beta + \cdots).$

But $\qquad \alpha + \beta + \cdots \doteq 0.$ [Art. 19, Theorem (*a*), Cor. 2]

Hence
$$\lim (x + y + \cdots) = a + b + \cdots$$
$$= \lim x + \lim y + \cdots.$$

Ex.
$$\lim .333 \cdots = \lim \left(\tfrac{3}{10} + \tfrac{3}{100} + \cdots\right) = \tfrac{1}{3},\ \lim .141414 \cdots = \tfrac{14}{99},$$
$$\lim (.333 \cdots + .141414 \cdots) = \lim (.474747 \cdots)$$
$$= \tfrac{47}{99} = \tfrac{1}{3} + \tfrac{14}{99}.$$

(*c*) *The limit of the product of a finite number of variables is the product of their limits.*

For, if $\qquad x \doteq a,\ y \doteq b,\ z \doteq c,$

then $\qquad x = a + \alpha,\ y = b + \beta,\ z = c + \gamma,$

in which $\qquad \alpha \doteq 0,\ \beta \doteq 0,\ \gamma \doteq 0.$

Hence $\qquad xyz = abc + bc\alpha + ca\beta + ab\gamma + a\beta\gamma + b\gamma\alpha + c\alpha\beta + \alpha\beta\gamma.$

$\therefore \lim xyz = abc$ $\qquad$ [Art. 19, Theorem (*a*), Cor. 2, Art. 20, Theorem (*a*)]

$\qquad = \lim x \cdot \lim y \cdot \lim z.$

N.B. Theorems (*a*), (*b*), (*c*), are true if one or more of the variables be replaced by constants.

Ex. $\qquad \lim (.333 \cdots \times .141414 \cdots) = \tfrac{1}{3} \cdot \tfrac{14}{99} = \tfrac{14}{297}.$

(*d*) *The limit of the quotient of two variables, x and y, whose limits are finite, is the quotient of their limits.*

Since $x = \frac{x}{y} \cdot y$, $\lim x = \lim \left(\frac{x}{y}\right) \cdot \lim y$ by Theorem (*c*).

$$\therefore \lim \left(\frac{x}{y}\right) = \frac{\lim x}{\lim y}.$$

Ex. $$\lim \frac{.333 \cdots}{.1414 \cdots} = \frac{\lim .333 \cdots}{\lim .1414 \cdots} = \tfrac{1}{3} \div \tfrac{14}{99} = \tfrac{33}{14}.$$

(*e*) *The order of an infinitesimal is not altered by adding or subtracting another infinitesimal of higher order.*

Let α be the standard infinitesimal, and β and γ be infinitesimals of orders m and n respectively, and n be greater than m. Now

$$\frac{\beta + \gamma}{\alpha^m} = \frac{\beta}{\alpha^m} + \frac{\gamma}{\alpha^m};$$

hence $$\lim \frac{\beta + \gamma}{\alpha^m} = \lim \frac{\beta}{\alpha^m} + \lim \frac{\gamma}{\alpha^m}. \qquad \text{[Theorem (b)]}$$

But $$\lim \frac{\gamma}{\alpha^m} = 0,$$

since $\gamma \div \alpha^m$ is an infinitesimal of order $n - m$, and n is greater than m.

$$\therefore \lim \frac{\beta + \gamma}{\alpha^m} = \lim \frac{\beta}{\alpha^m}.$$

NOTE. The order of an infinitesimal is not altered by adding an infinitesimal of the same order, but it may be altered by subtracting one of the same order. *E.g.* if $\beta = 2\alpha^3 + 3\alpha^4$, $\gamma = 2\alpha^3 - 3\alpha^6$, $\delta = \alpha^3 - \alpha^5$, then $\beta + \gamma = 4\alpha^3 + 3\alpha^4 - 3\alpha^6$, which is of the third order; $\beta - \gamma = 3\alpha^4 + 3\alpha^6$, which is of the fourth order; $\beta - \delta = \alpha^3 + 3\alpha^4 + \alpha^5$, which is of the third order.

Ex. Show that if x and y are two variables, and limit $(x \div y) = 1$, then $x - y$ is infinitesimal with respect to both x and y.

21. Fundamental theorems of the calculus.

A. **The limit of the quotient of any two variables, x and y, is not altered by adding to them any two numbers, say α and β, which are infinitesimal to x and y respectively;**

that is, $$\lim \frac{x + \alpha}{y + \beta} = \lim \frac{x}{y}, \text{ when } \frac{\alpha}{x} \doteq 0 \text{ and } \frac{\beta}{y} \doteq 0.$$

For $$\frac{x + \alpha}{y + \beta} = \frac{x}{y} \cdot \frac{1 + \frac{\alpha}{x}}{1 + \frac{\beta}{y}}.$$

$$\therefore \lim \frac{x+\alpha}{y+\beta} = \lim \frac{x}{y} \cdot \lim \frac{1+\dfrac{\alpha}{x}}{1+\dfrac{\beta}{y}} = \lim \frac{x}{y}.$$

[Art. 20, Theorems (*a*), (*c*), (*d*).]

NOTE 1. This is sometimes called the fundamental theorem of the *differential calculus*, as it is frequently used in that branch of the infinitesimal calculus. See Art. 22, Notes 1, 2.

Ex. 1. $$\text{Lim}_{a \doteq 0} \frac{2a^3 + 7a^6}{3a^3 + 5a^7} = \lim_{a \doteq 0} \frac{2a^3}{3a^3} = \tfrac{2}{3}.$$

Ex. 2. $$\text{Lim}_{\Delta x \doteq 0,\ \Delta y \doteq 0} \frac{2x + \Delta x}{2y + \Delta y} = \frac{x}{y}.$$ [See Ex. 3 (*a*), Art. 4.]

***B*. If the limit of the sum of any number of infinitesimals of the same sign be finite, this limit is not altered when any infinitesimal is replaced by another the limit of whose ratio to the first infinitesimal is unity.**

Let there be a set of any number of infinitesimals, $a_1, a_2, \cdots, a_n$, whose sum approaches a finite limit as n becomes infinitely great. Let $\beta_1, \beta_2, \cdots, \beta_n$, be another set of infinitesimals, such that

$$\lim \frac{\beta_1}{a_1} = 1,\ \lim \frac{\beta_2}{a_2} = 1,\ \cdots,\ \lim \frac{\beta_n}{a_n} = 1. \tag{1}$$

According to (1),

$$\frac{\beta_1}{a_1} = 1 + \epsilon_1,\ \frac{\beta_2}{a_2} = 1 + \epsilon_2,\ \cdots,\ \frac{\beta_n}{a_n} = 1 + \epsilon_n,$$

in which $$\epsilon_1 \doteq 0,\ \epsilon_2 \doteq 0,\ \cdots,\ \epsilon_n \doteq 0.$$

Then $\beta_1 = a_1 + \epsilon_1 a_1,\ \beta_2 = a_2 + \epsilon_2 a_2,\ \cdots,\ \beta_n = a_n + \epsilon_n a_n$;

and $\beta_1 + \beta_2 + \cdots + \beta_n = a_1 + a_2 + \cdots + a_n + (\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_n a_n)$.

$\therefore\ (\beta_1 + \beta_2 + \cdots + \beta_n) - (a_1 + a_2 + \cdots + a_n) = \epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_n a_n.$

$$\therefore\ \lim (\beta_1 + \beta_2 + \cdots + \beta_n) - \lim (a_1 + a_2 + \cdots + a_n) = \lim (\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_n a_n).$$

But $$\lim (\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_n a_n) = 0.$$

For let η be the numerically greatest of the ϵ's, then

$$(\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_n a_n) < (a_1 + a_2 + \cdots + a_n)\,\eta.$$

Now $\lim(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$ is finite, by hypothesis, and $\lim \eta = 0$; hence
$$\lim(\alpha_1 + \alpha_2 + \cdots + \alpha_n)\,\eta = 0,$$
and accordingly, $\lim(\epsilon_1\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n) = 0.$

Hence $\lim(\beta_1 + \beta_2 + \cdots + \beta_n) = \lim(\alpha_1 + \alpha_2 + \cdots + \alpha_n).$

NOTE 2. This is sometimes called the fundamental theorem of the *integral calculus*, as it is often used in that branch of infinitesimal calculus.

NOTE 3. A simple proof of B, depending on a theorem on fractions, is given in Gibson's *Calculus*, page 198.

NOTE 4. **References for collateral reading on infinitesimals.** McMahon and Snyder, *Differential Calculus* (American Book Co.), Arts. 1–15; J. J. Hardy, *Infinitesimals and Limits* (Chemical Publishing Co., Easton, Pa.), pamphlet 22 pages; Gibson, *Elementary Treatise on the Calculus*, Arts. 86, 87; Byerly, *Diff. Cal.*, Chap. X.

22. The derivative of a function of one variable. Suppose that the function
$$f(x)$$
denotes a continuous function of x. Let x receive an increment Δx; then the function becomes
$$f(x + \Delta x). \tag{a}$$
Hence the corresponding increment of the function is
$$f(x + \Delta x) - f(x). \tag{b}$$
This may be written $\Delta[f(x)]$.

The ratio of this increment of the function to the increment of the variable is
$$\frac{f(x + \Delta x) - f(x)}{\Delta x}; \quad i.e.\ \frac{\Delta[f(x)]}{\Delta x}. \tag{c}$$
The limit of this ratio when Δx approaches zero, i.e.
$$\lim_{\Delta x \doteq 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{or} \quad \lim_{\Delta x \doteq 0} \frac{\Delta f(x)}{\Delta x}, \tag{d}$$
is called **the derived function of $f(x)$** with respect to x; or **the derivative** (or **the derivate**) of $f(x)$ with respect to x; or **the x-derivative of $f(x)$**. It is also called *the differential coefficient of $f(x)$*, a name which is explained in Art. 27.

If y also be used to denote the function, that is, if

$$y = f(x),$$

then if x receive an increment Δx, y will receive a corresponding increment (positive or negative), which may be denoted by Δy, *i.e.*

$$y + \Delta y = f(x + \Delta x).$$

Hence $$\Delta y = f(x + \Delta x) - f(x);$$

and $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \qquad (e)$$

$$\therefore \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \qquad (f)$$

The process of finding the derivative of a function is called **differentiation.** This process is a perfectly general one, as indicated in steps (*a*), (*b*), (*c*), and (*d*). It may be described in words, thus:

(1) Give the independent variable an increment;

(2) Find the corresponding increment of the function;

(3) Write the ratio of the increment of the function to the increment of the variable.

(4) Find the limit of this ratio as the increment of the variable approaches zero.

For a slightly different description of the process of differentiation, see Note 4.

NOTE 1. To differentiate a function (*i.e.* to find its derivative) is one of the *three* main problems of the infinitesimal calculus; and is the main problem of that branch which is called "*the differential calculus.*"

NOTE 2. The other two main problems of the infinitesimal calculus (see Arts. 27 *a*, 94) are the main problems of that branch called "*the integral calculus.*" It may be said here that while the differential calculus solves the problem, "when the function is given, to find the derivative," on the other hand the integral calculus solves as one of its two main problems the inverse problem, namely, "when the derivative is given, to find the function."

EXAMPLES.

1. Find the derivative of x^3 with respect to x.

Here $$f(x) = x^3. \qquad \text{(See Fig., p.412.)}$$

Let x receive an increment Δx;

then $$f(x + \Delta x) = (x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

$$\therefore f(x + \Delta x) - f(x) = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

$$\therefore \frac{f(x + \Delta x) - f(x)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2.$$

$$\therefore \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 3x^2.$$

If y be used to denote the function, thus $y = x^3$, then the first members of these equations will be successively, y, $y + \Delta y$, Δy, $\frac{\Delta y}{\Delta x}$, $\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x}$.

NOTE 3. It should be observed that the expression (*c*) depends both on the value of x and the value of Δx, and, in general, contains terms that vanish with Δx, as exemplified in Ex. 1. (This is shown clearly in Art. 176.) On the other hand, *the value of the derivative depends on the value which x has when it receives the increment, and on that alone.* For this reason, the derivative of a function is often called *the derived function.* For instance, in Ex. 1, if $x = 2$, the value of the derivative is 12; if $x = 6$, the value of the derivative is 108. Compare Exs. in Arts. 3, 4. (It is probably now apparent to the beginner that the process used in the problems in Arts. 3, 4, was nothing more or less than differentiation.)

NOTE 4. Sometimes Δx is called the *difference of the variable x*, (*b*) is called the corresponding *difference of the function*, and (*c*) is called the *difference-quotient of the function.* The process of differentiation may then be described, thus: (1) Make a difference in the independent variable; (2) Calculate the corresponding difference made in the function; (3) Write the ratio of the difference in the function to the difference in the variable; (4) Determine the limiting value of this ratio when the difference in the variable approaches zero as a limit.

2. Find the derivatives, with respect to x, of x, $2x$, $3x$, ax, x^2, $7x^2$, $11x^2$, bx^2, x^3, $5x^3$, $13x^3$, and cx^3.

Ans. 1, 2, 3, a, $2x$, $14x$, $22x$, $2bx$, $3x^2$, $15x^2$, $39x^2$, $3cx^2$.

3. Calculate the values of these functions and the values of their derivatives, when $x = 1$, $x = 2$, $x = 3$.

4. Find the derivatives, with respect to x, of: (*a*) $x^2 + 2$, $x^2 - 7$, $x^2 + k$; (*b*) $x^3 + 7$, $x^3 - 9$, $x^3 + c$.

5. Differentiate x^4, $x^2 + 4x - 5$, $\frac{1}{x}$, $\frac{2}{x} - 3x + 2x^2$, with respect to x.

6. Find the derivatives, with respect to t, of $3t^2$, $4t^3 - 8t + \frac{3}{t}$.

7. Differentiate y^6, $\frac{3}{4}y^2 - 8y - \frac{7}{y}$, with respect to y.

8. Show that, if n is a positive integer, the derivative of x^n with respect to x, is nx^{n-1}.

NOTE 5. The result in Ex. 8, as will be seen later, *is true for all constant values of n.*

9. Assuming the result in Ex. 8, apply it to solve Exs. 4–7.

NOTE 6. In order that a function may be *differentiable* (*i.e.* have a derivative), it must be continuous; all continuous functions, however, are not differentiable. For remarks on this topic, see Echols, *Calculus*, Art. 30. For an example of a continuous function which has nowhere a determinate derivative, see Echols, *Calculus*, Appendix, Note 1, or Harkness and Morley, *Theory of Functions*, § 65.

23. Notation. There are various ways of indicating the derivative of a function of a single variable. (In what follows, the independent variable is denoted by x. In the case of other variables the symbols are similar to those now to be described for functions of x.)

(*a*) This symbol is often used to denote (*d*) Art. 22, viz.

$$f'(x). \qquad A$$

Thus the derivatives (or derived functions) of $F(x)$, $\phi(y)$, $f(t)$, $f_1(z)$, with respect to x, y, t, and z, respectively, are denoted by $F'(x)$, $\phi'(y)$, $f'(t)$, $f_1'(z)$. These are sometimes read "the F-prime function of x," etc.

(*b*) If y is used to denote the function of x (see Art. 22), the derivative of y with respect to x is frequently indicated by the symbol

$$y'. \qquad B$$

This is often read "y-prime"; but it is better to say "derivative of y."

(*c*) The x-derivative of $f(x)$ is also indicated by the symbol

$$\frac{d}{dx}f(x) \qquad C; \text{ or by } \frac{d[f(x)]}{dx}. \qquad D$$

The brackets in D are usually omitted, and the symbol is written

$$\frac{df(x)}{dx}. \qquad E$$

Symbols C, D, and E should be read "the x-derivative of $f(x)$."

(*d*) When y denotes the function, the derivative (see Equation (*f*) Art. 22) is sometimes denoted by

$$\frac{d}{dx}(y) \qquad F; \text{ or by } \frac{d(y)}{dx}. \qquad G$$

The brackets in F and G are usually omitted, and the symbol for the derivative is written

$$\frac{dy}{dx}. \qquad H$$

This should be read for a while at least by beginners, "the derivative of y with respect to x," or more briefly "*the x-derivative of y.*" (Other phrases, *e.g.* "*dy* by *dx*," are common, but, unfortunately, are misleading.)

(*e*) In case (*d*) the operation of differentiation, and also its result, namely, the derivative, are alike indicated by the symbol

$$Dy. \qquad I$$

(*f*) Sometimes the independent variable x is shown in the symbol, thus

$$D_x y. \qquad J$$

NOTE 1. Mathematics deals with various notions, and it discusses these notions in a language of its own. In the study of any branch of mathematics, the student has *first* to clearly understand its fundamental notions, and *then* to learn the peculiar shorthand language, made up of signs and symbols and phrases, which has been in part invented, and in part adapted, by mathematicians. A striking instance of the great importance of mere notation is seen in arithmetic. To-day a young pupil can easily perform arithmetical operations which would have taxed the powers of the great Greek mathematicians. The one enjoys the advantage of the convenient Arabic notation* for numerals, the other was hampered by the clumsy notation of the Greeks.

NOTE 2. Symbols A and B, and also I and J, have this important quality, namely, they tend to make manifest the fact that *the derivative is a single quantity.* It is *not* the ratio of two things, but is the *limiting value* of a variable ratio. Symbols C and F have the quality that they indicate, in a way, the process (Art. 22) by which the derivative is obtained. The symbol $\frac{d}{dx}$ before a function indicates that the operation of differentiation with respect to x is to be performed on the function; it also serves to indicate the result of the operation. The symbols D and D_x,† in I and J, are simply abbreviations for the symbol $\frac{d}{dx}$.

* This should really be called *the Hindoo notation;* for the Arabs obtained it from the Hindoos. See Cajori, *History of Mathematics.*

† The symbol $D_x y$ is due to Louis Arbogaste (1759–1803), professor of mathematics at Strasburg. The symbol $\frac{dy}{dx}$ was devised by Leibnitz, and the symbol f', by Lagrange (1736–1813).

NOTE 3. Beginners in the calculus are liable to be misled by the symbols D, E, G, and H, especially by H. *The symbol* $\frac{dy}{dx}$ *does not denote a fraction;* it does not mean "the ratio of a quantity dy to a quantity dx." Such quantities are not in existence at the stage when $\frac{dy}{dx}$ is obtained. It should be thoroughly realized, and never forgotten, that $\frac{dy}{dx}$ is short for $\frac{d}{dx}(y)$, and that both these symbols are merely abbreviations for $\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}$ (see Eq. (f), Art. 22). Some one has remarked that the dy and dx in $\frac{dy}{dx}$ are merely "the ghosts of departed quantities"; but perhaps this is claiming too much for them.

24. The geometrical meaning and representation of the derivative of a function. Let $f(x)$ denote a function, and let the geometrical representation of the function, namely the curve

$$y = f(x), \tag{1}$$

be drawn.

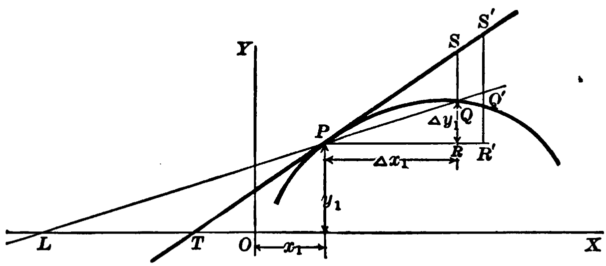

FIG. 10.

Let $P(x_1, y_1)$ and $Q(x_1 + \Delta x_1, y_1 + \Delta y_1)$ be two points on the curve. Draw the secant LPQ. Then

$$\tan PLX = \frac{\Delta y_1}{\Delta x_1}.$$

Now let secant LQ revolve about P until Q reaches P. Then the secant LP takes the position of the tangent TP, and the angle PLX becomes PTX; then, also, Δx_1 reaches zero.

Hence $$\tan PTX = \lim_{\Delta x_1 \doteq 0} \frac{\Delta y_1}{\Delta x_1}. \tag{2}$$

Now $P(x_1, y_1)$ is any point on the curve; hence, on letting (x, y), according to the usual custom, denote *any* point on the curve, and ϕ denote the angle made with the x-axis by the tangent at (x, y),

$$\tan \phi = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}. \tag{3}$$

The first member of (3) is the slope of the tangent at any point (x, y) on the curve $y = f(x)$, and the second member is the derivative of either member of (1). Hence $\frac{dy}{dx}$, **i.e. $f'(x)$, is the slope of the tangent at any point (x, y) on the curve $y = f(x)$.**

This principle has already been applied in the exercises in Art. 4.

Curve of slopes. If the graph of $f'(x)$ be drawn, that is, the curve $y = f'(x)$, it is called the *curve of slopes* of the curve $y = f(x)$. It is also called *the derived curve*, and sometimes *the differential curve of* $y = f(x)$. For instance, the curve of slopes of the curve $y = x^2$ is the line $y = 2x$. The curve of slopes is the geometrical representative of the derivative of the function; the measure of any of its ordinates is the same as the slope of $y = f(x)$ for the same value of x.

Ex. Sketch the graphs of the functions in Exs., Art. 22. Write the equations of these graphs. Give the equations of their curves of slopes, and sketch these curves. (Use the same axes for a curve and its curve of slopes.)

Note 1. Produce RQ (Fig. 10) to meet TP in S, produce PR to R, and draw $R'Q'S'$ parallel to RQ to meet the curve in Q' and TP in S'. Then

$$\frac{dy}{dx} \text{ or } f'(x) = \frac{RS}{PR} = \frac{R'S'}{PR'}.$$

Now, if $\Delta x_1 = PR$, $\frac{\Delta y_1}{\Delta x_1} = \frac{RQ}{PR}$; and if $\Delta x_1 = PR'$, $\frac{\Delta y_1}{\Delta x_1} = \frac{R'Q'}{PR'}$. Also,

$$\lim_{PR \doteq 0} \frac{RQ}{PR} = \frac{dy}{dx},$$

and likewise,

$$\lim_{PR' \doteq 0} \frac{R'Q'}{PR'} = \frac{dy}{dx}.$$

Note 2. Hereafter, in general investigations like the above, the symbol x will be used instead of x_1 to denote any particular value of x; and similarly in the case of other variables.

25. The physical meaning of the derivative of a function. Suppose that the value of a function, say s, depends upon time; *i.e.* suppose

$$s = f(t).$$

After an interval of time Δt, the function receives an increment Δs; and

$$s + \Delta s = f(t + \Delta t).$$

$$\therefore \ \Delta s = f(t + \Delta t) - f(t).$$

$$\therefore \ \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}. \tag{1}$$

$$\therefore \ \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} \left(i.e. \ \frac{ds}{dt} \right) = f'(t). \tag{2}$$

Since Δs is the change in the function during the time Δt, $\frac{\Delta s}{\Delta t}$ is the average rate of change of the function during that time. As Δt decreases, the average rate of change becomes more nearly equal to the rate of change at the time t, and can be made to differ from it by as little as one pleases, merely by decreasing Δt. Hence the second member of (2) is the actual rate of change at the time t. In words: **The derivative of a function with respect to the time is the rate of change of the function.**

If s denotes a varying distance along a straight line, then $\frac{ds}{dt}$ denotes the rate of change of this distance, i.e. *a velocity.*

(For discussions on speed and velocity see text-books on Kinematics and Dynamics, and Mechanics.)

Ex. Show that if $s = \frac{1}{2} gt^2$, then $\frac{ds}{dt} = gt$. (See Art. 3 *b*.)

NOTE. Newton called the calculus the *Method of Fluxions*. Variable quantities were called by him *fluents* or *flowing quantities*, and the rate of flow, *i.e.* the rate of increase of a variable, he called the *fluxion* of the fluent. Thus, if s and x are variable, $\frac{ds}{dt}$ and $\frac{dx}{dt}$ are their fluxions. Newton indicated these fluxions thus: $\dot{s}$, $\dot{x}$. This notation was adopted in England and held complete sway there until early in the last century, and the other notation, that of Leibnitz, prevailed on the continent. At last the continental notation was accepted in England. "The British began to deplore the very small progress that science was making in England as compared with its racing progress on the continent. In 1813 the 'Analytical Society' was formed at Cambridge. This was a small club established by George Peacock,

John Herschel, Charles Babbage, and a few other Cambridge students, to promote, as it was humorously expressed, the principles of pure 'D-ism,' that is, the Leibnitzian notation in the calculus against those of 'dot-age,' or of the Newtonian notation. The struggle ended in the introduction into Cambridge of the notation $\frac{dy}{dx}$, to the exclusion of the fluxional notation $\dot{y}$. This was a great step in advance, not on account of any great superiority of the Leibnitzian over the Newtonian notation, but because the adoption of the former opened up to English students the vast storehouses of continental discoveries. Sir William Thomson, Tait, and some other modern writers find it frequently convenient to use both notations." — Cajori, *History of Mathematics*, page 283.

26. General meaning of the derivative: the derivative is a rate. When a variable changes, a function of the variable also changes. A comparison of the change in the function with the causal change in the variable will determine *the rate of change of the function with respect to the variable.* The limit of the result of this comparison, as the change in the variable approaches zero, evidently gives this rate. But this limit has been defined as the derivative of the function with respect to the variable. Accordingly (see Art. 22, Note 1), *the main object of the differential calculus* may be said to be the determination of the rate of change of the function with respect to its argument.

NOTE 1. The rate of change of the function with respect to the variable may also be shown in a manner that explicitly involves the notion of time. In the case of the function y, when $y = f(x)$, let it be supposed that x receives a change Δx in a certain finite time Δt. Accordingly y will receive a change Δy in the same time Δt. Then, from the equation preceding (e), Art. 22,

$$\frac{\Delta y}{\Delta t} = \frac{f(x+\Delta x) - f(x)}{\Delta t} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \frac{\Delta x}{\Delta t}.$$

When Δt approaches zero, Δx also approaches zero. On letting Δt approach zero, this equation becomes (Art. 20, Th. c).

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}; \; i.e. \; \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}. \quad (1) \qquad \text{Whence, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \quad (2)$$

Result (1) can also, by a theorem on limits, Art. 20 (d), be derived from

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \div \frac{\Delta x}{\Delta t}.$$

Thus the derivative of a function with respect to a variable may be regarded as the ratio of the rate of change of the function to the rate of change of the variable.

NOTE 2. **References for collateral reading.** McMahon and Snyder, *Diff. Cal.*, Arts. 88, 89; Lamb, *Calculus*, Art. 33; Gibson, *Calculus*, Arts. 31–37, 51.

EXAMPLES.

1. A square plate of metal is expanding under the action of heat, and its side is increasing at a uniform rate of .01 inch per hour; what is the rate of increase of the area of the plate at the moment when the side is 16 inches long? At what rate is the area increasing 10 hours later?

Let x denote the side of the square and A denote its area. Then $A = x^2$. Now $\frac{\Delta A}{\Delta t} = \frac{\Delta A}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$; whence, $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$. $\therefore \frac{dA}{dt} = 2x \times .01$ sq. inches per hour $= .02\,x$ sq. inches per hour. Accordingly, at the moment when the side is 16 inches, the area of the plate is increasing at the rate of .32 sq. inches per hour. Ten hours later the side is 16.1 inches; the area of the plate is then increasing at the rate of .322 sq. inches per hour. The area of the square is increasing in square inches $2x$ times as fast as the side is increasing in linear inches.

2. In the case of a circular plate expanding under the action of heat, the area is increasing at any instant how many times as fast as the radius? If when the radius is 8 inches it is increasing .03 inches per second, at what rate is the area increasing? At what rate is the area increasing when the radius is 15 inches long?

3. The area of an equilateral triangle is expanding how many times as fast as each of its sides? At what rate is the area increasing when each side is 15 inches long and increasing at the rate of 2 inches a second? At what rate is the area increasing when each side is 30 inches long and increasing at the rate of 2 inches a second?

4. The volume of a spherical soap bubble is increasing how many times as fast as its radius? At what rate (cubic inches per second) is the volume increasing when the radius is half an inch and increasing at the rate of 3 inches per second? At what rate is the volume increasing when the radius is an inch?

5. A man 5 ft. 10 in. high walks directly away from an electric light 16 feet high at the rate of $3\frac{1}{2}$ miles per hour. How fast does the end of his shadow move along the pavement?

27. Differentials. If
$$y = f(x), \tag{1}$$
then, in accordance with notations *A* and *H*, in Art. 23,
$$\frac{dy}{dx} = f'(x). \tag{2}$$

Suppose that an arbitrary difference (*i.e.* change or increment) h may be made in the independent variable x, and let the product $f'(x) \cdot h$ be denoted by k; that is, let

$$k = f'(x) \cdot h. \tag{3}$$

(For instance, in Fig. 10, $RS = f'(x) \cdot PR$; here $h = PR$, and $k = RS$. Also, $R'S' = f'(x) \cdot PR'$; here $h = PR'$, and $k = R'S'$). Now, *let* h be written in the form dx, and the corresponding value of k be written dy. Then (3) is written

$$dy = f'(x)dx. \tag{4}$$

This, by (2), may be written

$$dy = \frac{dy}{dx} \cdot dx. \tag{5}$$

As used in Equation (4), dx is called *the differential of* x, dy is called *the differential of* y, and $f'(x)dx$ is called *the differential of* $f(x)$. Since $f'(x)$ is the coefficient of dx in the differential of $f(x)$, $f'(x)$ is frequently called *the differential coefficient of* $f(x)$. (See Art. 22.) The *defining* Equations (4) and (5) may be expressed in words: *The differential of a function* y *of an independent variable* x *is equal to the derivative of the function multiplied by the differential of the variable, the latter differential being merely an arbitrary increment* (*or difference*), *usually small, made in the variable.*

The letter d is used as the symbol for a differential; for example, the differential of $f(x)$ is written $df(x)$; thus $df(x) = f'(x)dx$.

NOTE 1. It is **highly important** to notice that in Equations (2) and (4), dy and dx are used in altogether different ways.* In (2) and (5), $\frac{dy}{dx}$ is used as a symbol for $\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}$; and it denotes the definite limiting value of an "indeterminate" form $\frac{0}{0}$. In (4) and in (5) on the extreme right dx is *not* zero (although it may happen to be, and usually is, a small quantity),† and the dy is such that the ratio $dy : dx$ is equal to $f'(x)$. For instance, in Fig. 10,

* In one respect this double use of dx and dy is unfortunate; for it tends to confuse beginners in calculus. Other notation is also used.

† Later on many examples will be found in which this dx is an infinitesimal.

$\frac{dy}{dx}$ of Equation (2) is tan SPR. As to Equations (4), (5), if $dx = PR$, then $dy = RS$, and if $dx = PR'$, then $dy = R'S'$. This shows that dy, in (4), is the increment of the ordinate *of the tangent* corresponding to an increment dx of the abscissa. The corresponding increment of the ordinate *of the curve* $y = f(x)$ [*i.e.* the increment of the function $f(x)$] in some cases can be found exactly by means of the equation of the curve, and in some cases can be found, in general only *approximately*, by means of a very important theorem in the calculus, namely, *Taylor's Theorem* (see Chap. XX.). Instances of the former are given below; instances of the latter are given in Art. 176.

NOTE 2. It should be clearly understood that, according to the preceding remarks, *cancellation of the dx's in* (5) *is impossible.*

N.B. For geometrie illustrations of derivatives and differentials see Art. 67.

EXAMPLES.

1. In the case of a falling body $s = \frac{1}{2} gt^2$ (see Art. 3); on denoting, as usual, the differential of the time by dt, ds, the corresponding differential of the distance is [Ex., Art. 3 (*b*)] $gtdt$; *i.e.* $ds = gtdt$. The actual change in s corresponding to the change dt in the time is [see Eq. (2), Art. 3 (*b*)] $gtdt + \frac{1}{2} g(dt)^2$.

2. In the curve $y = x^2$, $dy = 2\,x\,dx$. The actual change in y corresponding to the change dx in x is $2\,x\,dx + (dx)^2$. (See Eq. (1), Art. 4.) Thus if $x = 10$ and $dx = .001$, $dy = 2 \times 10 \times .001 = .02$. The actual change in the ordinate of the curve from $x = 10$ to $x = 10 + .001$ is $(10.001)^2 - 10^2$, *i.e.* .020001. This change may also be calculated as stated above, viz. $2 \times 10 \times .001 + (.001)^2$. The $dy = .02$ is the change in the ordinate of the tangent at $x = 10$ from $x = 10$ to $x = 10.001$ (see Note 1). (The student should use a figure with this example.)

3. Write the differentials of the functions in the Exs. in Art. 22.

4. Given that $y = x^3 - 4\,x^2$, find dy when $x = 4$ and $dx = .1$. Then find the change made in y when x changes from 4 to 4.1.

5. Given that $y = 2\,x^3 + 7\,x^2 - 9\,x + 5$, find dy when $x = 5$ and $dx = .2$. Then find the change made in y when x changes from 5 to 5.2.

NOTE 3. It is evident from these examples that the differential of a function is an approximation to the change in the function caused by a differential change in the variable; and that the smaller the differential of the variable, the closer is the approximation. When the differential varies and approaches zero it becomes an infinitesimal.

Ex. Calculate the differentials of the areas in Ex. 2, Art. 26, when the differential of the radius is .1 inch.

Ex. Calculate the differentials of the areas of the triangles in Ex. 3, Art. 26, when the differential of the side is .1 inch.

NOTE 4. It may be remarked here that in problems involving the use of the differential calculus derivatives more frequently occur, and in problems in integral calculus differentials (viz. infinitesimal differentials) are more in evidence.

NOTE 5. **References for collateral reading.** Gibson, *Calculus*, § 60; Lamb, *Calculus*, Arts. 57, 58.

27 *a*. Anti-derivatives and anti-differentials. In Arts. 22 and 27 the derivative and the differential of a function have been defined, and a general method of deducing them from the function has been described. With respect to the derivative and the differential the function is called an *anti-derivative* and an *anti-differential* respectively. Thus, if the function is x^2, the x-derivative and the x-differential are $2x$ and $2xdx$ respectively; on the other hand, x^2 is said to be an anti-derivative of $2x$ and an anti-differential of $2xdx$. To find the anti-derivatives and the anti-differentials of a given expression is one of the two main problems of the integral calculus. (See Art. 22, Notes 1, 2, and Arts. 94, 96, 97.)

NOTE. **Reference for collateral reading.** Perry, *Calculus for Engineers*, Arts. 12–24, 28, 36.

CHAPTER IV.

DIFFERENTIATION OF THE ORDINARY FUNCTIONS.

28. In this chapter the derivatives of the ordinary functions of elementary mathematics are obtained by the fundamental and general method described in Art. 22. Since these derivatives are frequently employed, a ready knowledge of them will prevent stumbling and thus make the subsequent work in calculus much simpler and easier; just as a ready command of the sums and products of a few numbers facilitates arithmetical work. Accordingly *these derivatives* should be *tabulated* by the student and *memorized.*

N.B. The beginner is earnestly recommended to try to derive these results for himself. For a synopsis of the chapter see Table of Contents.

GENERAL RESULTS IN DIFFERENTIATION.

29. The derivative of the sum of a function and a constant, namely, $\phi(x)+c$.

Put $$y = \phi(x) + c.$$

Let x receive an increment Δx; consequently y receives an increment, Δy say. That is,

$$y + \Delta y = \phi(x + \Delta x) + c.$$

$$\therefore\ \Delta y = \phi(x + \Delta x) + c - [\phi(x) + c]$$

$$= \phi(x + \Delta x) - \phi(x).$$

$$\therefore\ \frac{\Delta y}{\Delta x} = \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}.$$

Let Δx approach zero as a limit; then

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\phi(x+\Delta x) - \phi(x)}{\Delta x};$$

i.e. $$\frac{dy}{dx} = \phi'(x);$$

i.e. $$\frac{d}{dx}[\phi(x) + c] = \frac{d}{dx}[\phi(x)]. \tag{1}$$

Hence, *if constant* **terms** *appear in a function, they may be neglected when the function is differentiated.*

If u be used to denote $\phi(x)$, result (1) can be expressed:

$$\frac{d}{dx}(u+c) = \frac{du}{dx}. \tag{2}$$

COR. 1. It follows from (1) that **the derivative of a constant is zero.** This may also be derived thus: If $y = c$ a constant, then $y + \Delta y = c$; and, accordingly, $\Delta y = 0$. Hence, $\frac{\Delta y}{\Delta x} = 0$ for all values of Δx; hence, $\frac{dy}{dx}$, *i.e.* $\frac{d}{dx}(c)$, is zero.

COR. 2. If two functions differ by a constant, they have the same derivative.

From (2) and Art. 27, $d(u+c) = du$.

NOTE 1. In geometry $y = c$ is the equation of a straight line parallel to the axis of x and at a distance c from it. The slope of this line is zero; this is in accord with Cor. 1.

NOTE 2. The curves $y = \phi(x) + c$, in which c is an arbitrary constant (Art. 10), can be obtained by moving the curve $y = \phi(x)$ in a direction parallel to the y-axis. The result (1) shows that for the same value of the abscissa, the slope $\frac{dy}{dx}$ is the same for all the curves. See Figs. 11, 12, below.

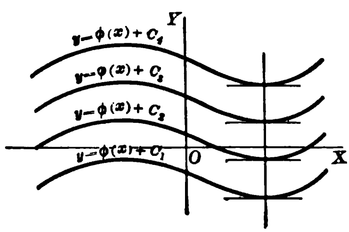

FIG. 11.

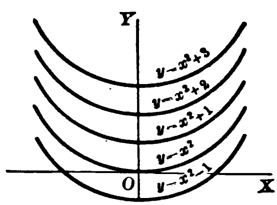

FIG. 12.

NOTE 3. The converse of Cor. 1 is also true; namely, *if the derivative of a quantity is zero, the quantity is a constant.*

Ex. Show this geometrically. (See Art. 24.)

NOTE 4. The converse of Cor. 2 is also true; namely, *if two functions have the same derivative, the functions differ only by an arbitrary constant.* (By the *same* derivative is meant *the same expression* in the variable and the fixed constants.) For let $\phi(x)$ and $F(x)$ denote the functions, and put

$$y = \phi(x) - F(x).$$

By hypothesis, $$Dy = \phi'(x) - F'(x) = 0.$$

Hence, by Note 3, $$y = c;$$

and accordingly, $$\phi(x) = F(x) + c.$$

Ex. Show this geometrically.

NOTE 5. If $\frac{dy}{dx} = \phi'(x)$, then $y = \phi(x) + c$, in which c denotes any constant. Hence $\phi(x) + c$ is a *general* expression for all the functions whose derivatives are $\phi'(x)$. Functions such as $\phi(x) + 1$, $\phi(x) - 3$, obtained by giving particular values to c, are *particular* functions having the same derivative $\phi'(x)$.

Note 6. Notes 4 and 5 come to this: **The anti-derivative of a function is indefinite, so far as an arbitrary additive constant is concerned.**

30. The derivative of the product of a constant and a function, say $c\phi(x)$.

Put $$y = c\phi(x).$$

Let x receive an increment Δx; consequently y receives an increment, Δy say.

That is, $$y + \Delta y = c\phi(x + \Delta x).$$

$$\therefore \Delta y = c[\phi(x + \Delta x) - \phi(x)].$$

$$\therefore \frac{\Delta y}{\Delta x} = c\left[\frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}\right].$$

$$\therefore \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} c\left[\frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}\right];$$

i.e. $$\frac{dy}{dx} = c\phi'(x);$$

i.e. $$\frac{d}{dx}[c\phi(x)] = c\phi'(x). \tag{1}$$

That is, ***the derivative of the product of a constant and a function is the product of the constant and the derivative of the function.***

If $\phi(x)$ be denoted by u, then (1) is written

$$\frac{d}{dx}(cu) = c\frac{du}{dx}. \tag{2}$$

In particular, if $u = x$, $\frac{d}{dx}(cx) = c$.

From the above and the definition in Art. 27, $d[c\phi(x)] = cd[\phi(x)]$, $d(cu) = c\,du$, $d(cx) = c\,dx$.

Ex. See Exs., Art. 22.

31. The derivative of the sum of a finite number of functions, say $\phi(x) + F(x) + \cdots$.

Put $$y = \phi(x) + F(x) + \cdots.$$

Then, on giving x an increment Δx (as in Arts. 29, 30),

$$y + \Delta y = \phi(x + \Delta x) + F(x + \Delta x) + \cdots.$$

$$\therefore \Delta y = \phi(x + \Delta x) - \phi(x) + F(x + \Delta x) - F(x) + \cdots.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} + \frac{F(x + \Delta x) - F(x)}{\Delta x} + \cdots.$$

Hence, on letting Δx approach zero,

$$\frac{dy}{dx} = \frac{d}{dx}\phi(x) + \frac{d}{dx}F(x) + \cdots; \tag{1}$$

i.e. $$\frac{d}{dx}[\phi(x) + F(x) + \cdots] = \phi'(x) + F'(x) + \cdots. \tag{2}$$

That is, *the derivative of a sum of a finite number of functions is the sum of their derivatives.*

If the functions be denoted by $u, v, w, \cdots$, *i.e.* if

$$y = u + v + w + \cdots,$$

the result (1) may be expressed thus:

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \cdots.$$

From this and Art. 27,

$$dy = du + dv + dw + \cdots.$$

NOTE 1. The differentiation of the sum of an *infinite* number of functions is discussed in Art. 173.

In working the following exercise the result of Ex. 8, Art. 22, may be used.

Ex. Find the derivatives of

$$2x^3 + 7x^2 - 10x + 11,\quad x^2 - 17x + 10,\quad -x^2 + 21x - 5.$$

32. The derivative of the product of two functions, say $\phi(x)F(x)$.

Put $$y = \phi(x)F(x).$$

Then, on giving x an increment Δx,

$$y + \Delta y = \phi(x + \Delta x)F(x + \Delta x).$$

$$\therefore \Delta y = \phi(x + \Delta x)F(x + \Delta x) - \phi(x)F(x).$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\phi(x + \Delta x)F(x + \Delta x) - \phi(x)F(x)}{\Delta x}. \tag{1}$$

On letting $\Delta x \doteq 0$, the second member takes the form $\frac{0}{0}$. In order to evaluate this form, introduce $\phi(x + \Delta x)F(x) - \phi(x + \Delta x)F(x)$ in the numerator of this member.* Then, on combining and arranging terms, (1) becomes

$$\frac{\Delta y}{\Delta x} = \phi(x + \Delta x)\frac{F(x + \Delta x) - F(x)}{\Delta x} + F(x)\frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}.$$

Hence, on letting Δx approach zero,

$$\frac{dy}{dx} = \phi(x)F'(x) + F(x)\phi'(x). \tag{2}$$

That is: *The derivative of the product of two functions is equal to the product of the first by the derivative of the second* plus *the product of the second by the derivative of the first.*

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* *Equally well*, $\phi(x)F(x + \Delta x) - \phi(x)F(x + \Delta x)$ may be thus introduced. The student should do this as an exercise.

If the functions be denoted by u and v, that is, if

$$y = uv,$$

then (2) may be expressed

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{3}$$

The derivative of the product of any finite number of functions can be obtained by an extension of (3). For example, if

$$y = uvw,$$

then, on regarding vw as a single function,

$$\begin{aligned} \frac{dy}{dx} &= (vw)\frac{du}{dx} + u\frac{d}{dx}(vw) \\ &= vw\frac{du}{dx} + u\left(w\frac{dv}{dx} + v\frac{dw}{dx}\right) \\ &= vw\frac{du}{dx} + wu\frac{dv}{dx} + uv\frac{dw}{dx}. \end{aligned} \tag{4}$$

Similarly, if $y = uvwz$,

$$\frac{dy}{dx} = vwz\frac{du}{dx} + uwz\frac{dv}{dx} + uvz\frac{dw}{dx} + uvw\frac{dz}{dx}. \tag{5}$$

In general: *In order to find the derivative of a product of several functions, multiply the derivative of each function in turn by all the other functions, and add the results.*

NOTE. Another way of obtaining (5) is given in Art. 39 (a).

The differential of the product of two functions. If

$$y = uv,$$

then, from (3) and the definition in Art. 27, it follows that

$$dy = u\frac{dv}{dx}dx + v\frac{du}{dx}dx. \tag{6}$$

But, by Art. 27, $\frac{dv}{dx}dx = dv$, and $\frac{du}{dx}dx = du$.

Hence, (6) may be written

$$d(uv) = udv + vdu. \tag{7}$$

Similarly, if $$y = uvw,$$
it follows from (4) that $dy = vwdu + wudv + uvdw.$

On division by uvw, this takes the form

$$\frac{d(uvw)}{uvw} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w}. \tag{8}$$

Ex. 1. Write dy in forms (7) and (8), when $y = uvwz$.

Ex. 2. Differentiate $(x^3 + 1)(x^2 - 2x + 7)$ by the above method; then expand this product and differentiate, and show that the results are the same.

Ex. 3. Treat the following functions as indicated in Ex. 2:

$$x^2(x-1)(x^3+4), \quad (ax^2+bx+c)(lx+m).$$

Ex. 4. Write the differentials of the functions in Exs. 2, 3.

33. The derivative of the quotient of two functions, say $\phi(x) \div F(x)$.

Put $$y = \frac{\phi(x)}{F(x)}.$$

Then, on proceeding as in Arts. 29–32,

$$y + \Delta y = \frac{\phi(x+\Delta x)}{F(x+\Delta x)}.$$

$$\therefore \Delta y = \frac{\phi(x+\Delta x)}{F(x+\Delta x)} - \frac{\phi(x)}{F(x)}$$

$$= \frac{\phi(x+\Delta x)F(x) - \phi(x)F(x+\Delta x)}{F(x)F(x+\Delta x)}.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\phi(x+\Delta x)F(x) - \phi(x)F(x+\Delta x)}{F(x)F(x+\Delta x)\Delta x}. \tag{1}$$

On letting $\Delta x \doteq 0$, the second member takes the form $\frac{0}{0}$. In order to evaluate this form, introduce

$$F(x)\phi(x) - F(x)\phi(x)$$

in the numerator of this member. Then, on combining and arranging terms, (1) becomes

$$\frac{\Delta y}{\Delta x} = \frac{F(x)\left[\dfrac{\phi(x+\Delta x)-\phi(x)}{\Delta x}\right] - \phi(x)\left[\dfrac{F(x+\Delta x)-F(x)}{\Delta x}\right]}{F(x)F(x+\Delta x)}.$$

Hence, on letting Δx approach zero,

$$\frac{dy}{dx}=\frac{F(x)\phi'(x)-\phi(x)F'(x)}{[F(x)]^2}. \tag{1}$$

That is: *If one function be divided by another, then the derivative of the fraction thus formed is equal to the product of the denominator by the derivative of the numerator* minus *the product of the numerator by the derivative of the denominator, all divided by the square of the denominator.*

If the functions be denoted by u and v; that is, if

$$y=\frac{u}{v},$$

then (1) has the form

$$\frac{dy}{dx}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}. \tag{2}$$

The differential of the quotient of two functions. If $y=\dfrac{u}{v}$, then from (2) and the definition in Art. 27,

$$dy=\frac{v\dfrac{du}{dx}dx-u\dfrac{dv}{dx}dx}{v^2}. \tag{3}$$

But, by Art. 27, $\dfrac{du}{dx}dx=du$ and $\dfrac{dv}{dx}dx=dv$. Hence (3) may be written

$$dy=\frac{v\,du-u\,dv}{v^2}. \tag{4}$$

NOTE. The derivative (1), or (2), can also be obtained by means of Art. 32. For if $y=\dfrac{u}{v}$, then $vy=u$. Whence $v\dfrac{dy}{dx}+y\dfrac{dv}{dx}=\dfrac{du}{dx}$. From this $\dfrac{dy}{dx}=\dfrac{1}{v}\dfrac{du}{dx}-\dfrac{y}{v}\dfrac{dv}{dx}$, which reduces to the form in (2) on substituting $\dfrac{u}{v}$ for y.

Ex. 1. Find the derivatives and the differentials of

$$\frac{x^3}{3x^2-7x+2},\quad \frac{x^2+7}{x^3+8},\quad \frac{x-11}{2x^2-9x+3}.$$

Ex. 2. Calculate the differentials of the functions in Ex. 1 when $x=2$ and $dx=.1$.

34. The derivative of a function of a function.

Suppose that $$y = \phi(u),$$

and that $$u = F(x),$$

and that the derivative of y with respect to x is required. (Here $\phi(u)$ and $F(x)$ are continuous functions.) The method which naturally comes first to mind, is to substitute $F(x)$ for u in the first equation, thus getting $y = \phi[F(x)]$, and then to proceed according to preceding articles. This method, however, is often more tedious and difficult than the one now to be shown.

Let x receive an increment Δx; accordingly, u receives an increment Δu, and y receives an increment Δy. Then

$$y + \Delta y = \phi(u + \Delta u).$$

$$\therefore \; \Delta y = \phi(u + \Delta u) - \phi(u).$$

$$\therefore \; \frac{\Delta y}{\Delta x} = \frac{\phi(u + \Delta u) - \phi(u)}{\Delta x}$$

$$= \frac{\phi(u + \Delta u) - \phi(u)}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Now Δx, Δu, Δy reach the limit zero together. Hence (Art. 20, Th. *c*) on letting Δx approach zero,

$$\frac{dy}{dx} = \frac{d}{du}[\phi(u)] \cdot \frac{du}{dx};$$

$$\textit{i.e.} \; \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{1}$$

NOTE. It should be clearly understood that the first member of (1) does not come, and cannot come, from the second member by cancellation of the du's. Cancellation is not involved at all.

Result (1), which may be expressed more emphatically (Art. 23),

$$\frac{d}{dx}(y) = \frac{d}{du}(y) \cdot \frac{d}{dx}(u), \tag{2}$$

is an important one and has frequent applications. It may be thus stated: *the derivative of a function with respect to a variable is equal to the product of the derivative of the function with respect to a second function and the derivative of the second function with respect to the first named variable.* (Here all the functions concerned are supposed to be continuous.)

From (1) and (2) it results that

$$\frac{d}{du}(y)=\frac{\frac{d}{dx}(y)}{\frac{d}{dx}(u)}, \text{ i.e. } \frac{dy}{du}=\frac{\frac{dy}{dx}}{\frac{du}{dx}}. \tag{3}$$

Relations (1) and (2), Note 1, Art. 26, are special applications of (1) [or (2) and (3)]. The showing of this is left as an exercise for the student.

Ex. 1. Explain why the du's in (1) may not be cancelled.

Ex. 2. Find $\frac{dy}{dx}$, given that $y = u^3$ and $u = x^2 + 1$.

Here $\frac{dy}{du} = 3u^2$, $\frac{du}{dx} = 2x$. $\therefore \frac{dy}{dx} = 6u^2x = 6x(x^2+1)^2$.

Ex. 3. Find $\frac{dy}{dx}$ when $y = 3u^2$ and $u = x^4 - 3x + 7$. Verify the result by the substitution method referred to at the beginning of the article.

Ex. 4. Find $\frac{dz}{dt}$ when $z = 2v^2 - 3v + 1$ and $v = 6t^2 + 1$. Verify the result by the substitution method.

Ex. 5. Show that *a function of a function is represented by a curve in space.* (See Echols, *Calculus*, Appendix, Note 2.)

35. The derivative of one variable with respect to another when both are functions of a third variable.

Let $x = F(t)$ and $y = \phi(t)$.

Now $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \div \frac{\Delta x}{\Delta t}$. Now Δt, Δx, and Δy reach the limit zero together.

Hence, Art. 20, Th. d, on letting Δt approach zero,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{1}$$

This result may also be derived as a special case of result (3), Art. 34. This is left as an exercise for the student.

Ex. 1. Find $\frac{dy}{dx}$ when $y = 3t^2 - 7t + 1$, and $x = 2t^3 - 13t^2 + 11t$.

Here $\frac{dy}{dt} = 6t - 7$, $\frac{dx}{dt} = 6t^2 - 26t + 11$. $\therefore \frac{dy}{dx} = \frac{6t-7}{6t^2-26t+11}$.

Ex. 2. Find $\frac{dy}{dx}$ when $x = 2t^2 + 17t - 1$ and $y = 3t^4 - 8t^2 + 9$.

Ex. 3. Find $\frac{du}{dv}$ when $u = 7x^4 - 3$ and $v = 3x^2 + 14x - 4$.

36. Differentiation of inverse functions. If y is a function of x, then x is a function of y; the second function is said to be *the inverse function* of the first. This is expressed by the following notation: If $y=f(x)$, then $x=f^{-1}(y)$. Examples of inverse notation have been met in trigonometry.

The equation $\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y}=1$ is always true. Accordingly (Art. 20, Th. *c*), $\frac{dy}{dx}\cdot\frac{dx}{dy}=1.$

Hence,
$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}.$$

DIFFERENTIATION OF PARTICULAR FUNCTIONS.

In the following articles **u denotes a continuous function of x,** and differentiation is made with respect to x. The letters a, n, $\cdots$, may denote constants.

N.B. ***It is advisable for the student to try to obtain the derivatives before having recourse to the book for help.***

A. Algebraic Functions.

37. Differentiation of u^n.

(*a*) *For n, a positive integer.*

Put
$$y=u^n;$$

i.e.
$$y=uuu\cdots \text{ to } n \text{ factors.}$$

$$\therefore\ \frac{dy}{dx}=u^{n-1}\frac{du}{dx}+u^{n-1}\frac{du}{dx}+\cdots \text{ to } n \text{ terms} \quad \text{(Art. 32)}$$
$$=nu^{n-1}\frac{du}{dx}.$$

In particular, $\frac{d}{dx}(x)=1$, and $\frac{d}{dx}(x^n)=nx^{n-1}$.

Ex. 1. Give the derivatives with respect to x of

u^2, $3u^4$, $7u^9$, x^8, $3x^4$, $7x^{12}$, $9x^8-17x^2+10x+40$.

Ex. 2. Find the x-derivative of $(2x+7)^{18}$.

On denoting this function by y, and putting u for $2x+7$, $y=u^{18}$. Hence

$$\frac{dy}{dx}=18\,u^{17}\frac{du}{dx}.$$

Now $\dfrac{du}{dx}=2$; hence $\dfrac{dy}{dx}=36\,u^{17}=36\,(2x+7)^{17}.$

The substitution u for $2x+7$ need not be explicitly made. For, if

$$y=(2x+7)^{18},$$

then
$$\frac{dy}{dx}=18\,(2x+7)^{17}\frac{d}{dx}(2x+7) \quad \text{(Art. 34)}$$
$$=36\,(2x+7)^{17}.$$

Ex. 3. Differentiate

$$(5x^2-10)^{24},\quad (3x^4+2)^{10},\quad (4x^2+5)^8(3x^4-2x+7)^5.$$

(*b*) *For n, a negative integer.* Let $n=-m$, and put $y=u^n$.

Then
$$y=u^{-m}=\frac{1}{u^m}.$$

$$\therefore\ \frac{dy}{dx}=\frac{u^m\cdot\dfrac{d}{dx}(1)-1\cdot\dfrac{d}{dx}(u^m)}{u^{2m}} \quad \text{(Art. 33)}$$

$$=\frac{-mu^{m-1}\dfrac{du}{dx}}{u^{2m}}=(-m)\,u^{(-m)-1}\frac{du}{dx}$$

$$=nu^{n-1}\frac{du}{dx}.$$

Ex. 4. Differentiate with respect to x,

$$u^{-2},\quad u^{-7},\quad u^{-11},\quad x^{-7},\quad 3x^{-5},\quad 17x^{-10},\quad (x^2-3)^{-4},\quad (3x^4+7)^{-5},$$
$$3x^5-7x^3+2-\frac{1}{x}+\frac{5}{x^2}-\frac{1}{9x^3}.$$

(*c*) *For n, a rational fraction.* Let $n=\dfrac{p}{q}$, in which p and q are integers.

Put $y=u^{\frac{p}{q}}$; then $y^q=u^p$.

On differentiating, $\qquad qy^{q-1}\dfrac{dy}{dx}=pu^{p-1}\dfrac{du}{dx}.$

$$\therefore\ \frac{dy}{dx}=\frac{p}{q}\,\frac{u^{p-1}}{y^{q-1}}\,\frac{du}{dx}=\frac{p}{q}\,\frac{u^{p-1}}{u^{\frac{p}{q}(q-1)}}\,\frac{du}{dx}=\frac{p}{q}\,u^{\frac{p}{q}-1}\frac{du}{dx}=nu^{n-1}\frac{du}{dx}.$$

Ex. 5. Find the x-derivatives of

$$\sqrt{u} \text{ (i.e. } u^{\frac{1}{2}}), \quad u^{-\frac{3}{4}}, \quad u^{\frac{7}{5}}, \quad \sqrt{x}, \quad x^{\frac{7}{2}}, \quad \sqrt{x^5}, \quad \sqrt{3x^2-5},$$

$$\sqrt[3]{2x^2+7x-3}, \quad \sqrt{2x+7}, \quad (3x-7)^{-\frac{8}{3}}, \quad 3x^2-7x^{\frac{1}{2}}+\frac{2}{x^{\frac{1}{2}}}+\frac{3}{x^{\frac{2}{3}}}-\frac{2}{7x^{\frac{2}{5}}}.$$

(*d*) *For n, an incommensurable number.* In this case it is also true that $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$. This is proved in Art. 39 (*b*).

Hence, for all constant values of n,

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \tag{1}$$

In particular, if $u = x$, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Ex. 6. Find the x-derivatives of

$$u^{\sqrt{2}}, \; x^{\sqrt{3}}, \; 5x^{\sqrt{7}}, \; (2x+5)^{\sqrt{5}}, \; (3x^2+7x-4)^{\sqrt{3}}.$$

Ex. 7. Write three functions which have x^3 for a derivative.

Ex. 8. Do as in Ex. 7 for the functions

$$x^5, \; \frac{1}{x^2}, \; \sqrt{x}, \; \sqrt{x^3}, \; \sqrt[3]{x}, \; 6x^4 - \frac{2}{x^2} - \frac{1}{\sqrt{x}}.$$

Ex. 9. Show that *the general form which includes all the functions that have x^n for the derivative, is* $\frac{x^{n+1}}{n+1} + c$, *in which c is an arbitrary constant.*

NOTE 1. The result (1) and the general results, Arts. 29–36, suffice for the differentiation of any algebraic function.

NOTE 2. Case (*a*) can also be treated as follows: Put $y = u^n$, and let x receive an increment Δx; then u and y receive increments Δu and Δy respectively. Then $y + \Delta y = (u + \Delta u)^n$. On expanding the second member by the binomial theorem, then calculating Δy and then $\frac{\Delta y}{\Delta x}$, and finally letting Δx approach zero, the result will be obtained.

NOTE 3. It is well to remember that $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

Ex. 10. Do the operations indicated in Note 2.

Ex. 11. Differentiate $\frac{x\sqrt{x^2+7}}{\sqrt[3]{x^2+2}}$. Find the value of the derivative when $x = 2$.

Put $$y = \frac{x(x^2+7)^{\frac{1}{2}}}{(x^2+2)^{\frac{1}{3}}}.$$

Then $$\frac{dy}{dx} = \frac{(x^2+2)^{\frac{1}{3}}\frac{d}{dx}[x(x^2+7)^{\frac{1}{2}}] - x(x^2+7)^{\frac{1}{2}}\frac{d}{dx}(x^2+2)^{\frac{1}{3}}}{(x^2+2)^{\frac{2}{3}}}.$$

On performing the differentiations indicated in the second member, and reducing, it is found that

$$\frac{dy}{dx} = \frac{4x^4+19x^2+42}{3(x^2+7)^{\frac{1}{2}}(x^2+2)^{\frac{4}{3}}}.$$

Hence, when $x = 2$,

$$\frac{dy}{dx} = 1.68, \text{ approximately.}$$

Ex. 12. Differentiate the following functions with respect to x:

$(2x-5)(x^2+11x-3)$, $ax^m + \dfrac{b}{x^n}$, $\dfrac{1+x^2}{1-x^2}$, $\dfrac{a-x}{a+x}$, $\sqrt{1+x^2}$, $\dfrac{3}{x^4} + 5\sqrt[3]{x} - 7x^5$, $\dfrac{\sqrt{1+x^2}}{x}$, $\dfrac{x}{\sqrt{a-bx^2}}$, $\dfrac{x^3}{(1-x^2)^{\frac{3}{2}}}$, $\sqrt{\dfrac{1+x}{1-x}}$, $(1+x^n)^m$, $(a+bx^3)^4$, $x^m(1-x)^n$, $(a+x)\sqrt{a-x}$.

Ex. 13. Find $\dfrac{dy}{dx}$ when $x^2y^3 + 2x + 3y = 5$. Here y is *an implicit function of x.* On differentiation of both members with respect to x,

$$x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x^2) + 2 + 3\frac{dy}{dx} = 0;$$

i.e. $$3x^2y^2\frac{dy}{dx} + 2xy^3 + 2 + 3\frac{dy}{dx} = 0.$$

From this $$\frac{dy}{dx} = -\frac{2(1+xy^3)}{3(1+x^2y^2)}.$$

Ex. 14. (*a*) Find $\dfrac{dy}{dx}$ when x and y are connected by the following relations: $y^3 + x^3 - 3axy = 0$; $x^4 + 2ax^2y - ay^3 = 0$; $7x^2y^2 + 2xy^3 - 3x^3y + 4x^2 - 8y^2 = 5$; $(a+y)^2(b^2-y^2) + (x+a)^2y^2 = 0$; $x^2 + y^2 = a^2$; $a^2y^2 + b^2x^2 = a^2b^2$. In the last case also obtain $\dfrac{dy}{dx}$ directly in terms of x.

(*b*) In the ellipse $3x^2 + 4y^2 = 7$, find the slope at the points $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$.

N.B. The following examples should all be worked by the beginner. They will serve to test and strengthen his grasp of the fundamental principles of the subject, and will give him exercise in making practical applications of his knowledge. For those who may not succeed in solving them

after a good endeavour, two examples are worked in the note at the end of the set.

Ex. 15. A ladder 24 feet long is leaning against a vertical wall. The foot of the ladder is moved away from the wall, along the horizontal surface of the ground and in a direction at right angles to the wall, at a uniform rate of 1 foot per second. Find the rate at which the top of the ladder is descending on the wall when the foot is 12 feet from the wall.

Ex. 16. Show that when the top of the ladder is 1 foot from the ground, the top is moving 575 times as fast as when the foot of the ladder is 1 foot from the wall.

Ex. 17. Find a curve whose slope at any point (x, y) is $2x$. Find a general equation that will include the equations of all such curves. Find the particular curve which passes through the point (1, 2).

Ex. 18. A man standing on a wharf is drawing in the painter of a boat at the rate of 4 feet a second. If his hands are 6 feet above the bow of the boat, how fast is the boat moving when it is 8 feet from the wharf?

Ex. 19. A man 6 feet high walks away at the rate of 4 miles an hour from a lamp post 10 feet high. At what rate is the end of his shadow increasing its distance from the post? At what rate is his shadow lengthening?

Ex. 20. A tangent to the parabola $y^2 = 16x$ intersects the x-axis at 45°. Find the point of contact.

Ex. 21. A ship is 75 miles due east of a second ship. The first sails west at the rate of 9 miles an hour, the second south at the rate of 12 miles an hour. How long will they continue to approach each other? What is the nearest distance they can get to each other?

Ex. 22. A vessel is anchored in 10 fathoms of water, and the cable passes over a sheave in the bowsprit which is 12 feet above the water. If the cable is hauled in at the rate of a foot a second, how fast is the vessel moving through the water when there are 20 fathoms of cable out?

Ex. 23. Sketch the curves $y^2 = 4x$ and $x^2 = 4y$, and find the angles at which they intersect. (If θ denotes the angle between lines whose slopes are m and n, $\tan\theta = (m - n) \div (1 + mn)$; see analytic geometry and plane trigonometry.)

Ex. 24. Sketch the curves $y^2 = 8x$ and $x^2 = 8y$, and find the angles at which they intersect.

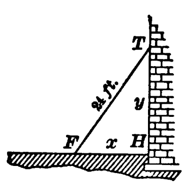

FIG. 13.

Note. Examples worked. Ex. 15. Let FT be the ladder in one of the positions which it takes during the motion, and let FH be the horizontal projection of FT. Let $FH = x$, and $HT = y$. Then

$$x^2 + y^2 = 576. \qquad (1)$$

Now x and y are varying with the time; the time-rate $\frac{dx}{dt}$ is given, and the time-rate $\frac{dy}{dt}$ is required. Differentiation of both members of (1) with respect to the time give

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0;$$

whence $$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}. \tag{2}$$

In this case, $\frac{dx}{dt} = 1$ foot per second, $x = 12$ feet, and, accordingly,

$$y = \sqrt{24^2 - 12^2} \text{ feet} = 12\sqrt{3} \text{ feet}.$$

$$\therefore \frac{dy}{dt} = -\frac{12}{12\sqrt{3}} \cdot 1 \text{ foot per second} = -.577 \text{ feet per second}.$$

The negative sign indicates that y decreases as x increases. It should be noticed that the result (2) is *general*, and that all particular solutions can be derived from it by substituting in it the particular values of x, y, and $\frac{dx}{dt}$.

Ex. 17. Find a curve whose slope at any point (x, y) is $2x$. Find a general equation that will include the equations of all such curves; and find the particular curve which passes through the point (1, 2).

Here $$\frac{dy}{dx} = 2x.$$

Hence $$y = x^2 + c, \tag{1}$$

in which c denotes any arbitrary *constant*. This is the *general* equation of all the curves having the slope $2x$. $\therefore$ $y = x^2 + 7$ is one of the curves; $y = x^2 - 5$ is another. If the point (1, 2) is on one of the curves (1), then $2 = 1 + c$; whence $c = 1$, and, accordingly, $y = x^2 + 1$ is the particular curve passing through (1, 2). As in Ex. 15 it is easier to find first the *general solution* of the problem in question, and therefrom to obtain any particular solution that may be required. Figure 12 shows some of these curves.

B. LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

38. Note. To find $\lim_{m \doteq \infty}\left(1 + \frac{1}{m}\right)^m$. This limit is required in what follows.

(*a*) *For m, a positive integer.* By the binomial theorem,

$$\left(1 + \frac{1}{m}\right)^m = 1 + m \cdot \frac{1}{m} + \frac{m \cdot m - 1}{1 \cdot 2} \cdot \frac{1}{m^2} + \frac{m \cdot m - 1 \cdot m - 2}{1 \cdot 2 \cdot 3} \cdot \frac{1}{m^3} + \cdots. \tag{1}$$

This can be put in the form

$$\left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{1\left(1 - \frac{1}{m}\right)}{2!} + \frac{1\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)}{3!} + \cdots. \tag{2}$$

On letting m approach infinity, and taking the limits, this becomes *

$$\lim_{m \doteq \infty}\left(1+\frac{1}{m}\right)^m = 1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \cdots$$

$$= 2.718281829 \cdots. \tag{3}$$

This constant number is always denoted by the symbol e.

(b) The result (3) is true for all infinitely great numbers, positive and negative, integral, fractional, and incommensurable. For the proof of (3) for all kinds of numbers, see Chrystal, *Algebra* (ed. 1889), Part II., Chap. XXV., § 13, Chap. XXVIII., §§ 1–3; McMahon and Snyder, *Diff. Cal.*, Art. 30, and Appendix, Note B; Gibson, *Calculus*, § 48.

NOTE ON e. The transcendental number e frequently presents itself in investigations in algebra (for instance, as the base of the natural logarithms, and in the theory of probability), in geometry, and in mechanics. The numbers e and π are perhaps the two most important numbers in mathematics. They are closely allied, being connected by the very remarkable relation $e^{i\pi} = -1$,† which was discovered by Euler. See references above, and Klein, *Famous Problems* (referred to in footnote, Art. 8), pages 55–67.

39. Differentiation of $\log_a u$.

Put $$y = \log_a u,$$

and let x receive an increment Δx; then u and y consequently receive increments Δu and Δy respectively.

Then $$y + \Delta y = \log_a (u + \Delta u).$$

$$\therefore\ \Delta y = \log_a (u + \Delta u) - \log_a u$$

$$= \log_a \left(\frac{u+\Delta u}{u}\right) = \log_a \left(1 + \frac{\Delta u}{u}\right).$$

$$\therefore\ \frac{\Delta y}{\Delta x} = \log_a \left(1 + \frac{\Delta u}{u}\right) \cdot \frac{1}{\Delta x}.$$

On introducing $\frac{1}{u} \cdot \frac{u}{\Delta u} \cdot \Delta u$ in the second member,

$$\frac{\Delta y}{\Delta x} = \frac{1}{u} \cdot \frac{u}{\Delta u} \log_a \left(1 + \frac{\Delta u}{u}\right) \cdot \frac{\Delta u}{\Delta x} = \frac{1}{u} \log_a \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} \cdot \frac{\Delta u}{\Delta x}.$$

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* This conclusion is properly reached only after a more rigorous investigation than is here attempted. (See Arts. 167–171.)

† See Art. 179.

From this, on letting Δx approach zero and remembering that Δu and Δy approach zero with Δx, it follows by Arts. 22, 23, 38, that

$$\frac{dy}{dx} = \frac{1}{u} \cdot \log_a e \cdot \frac{du}{dx};$$

i.e.
$$\frac{d}{dx}(\log_a u) = \frac{1}{u} \cdot \log_a e \cdot \frac{du}{dx}.$$

If $u = x$, then
$$\frac{d}{dx}(\log_a x) = \frac{1}{x} \cdot \log_a e.$$

If $a = e$, then
$$\frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}.$$

If $u=x$, and $a=e$, then $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}.$

NOTE. When e is the base it is usual not to indicate it in writing the logarithm.

Ex. 1. Find the derivatives of $\log_a (3x^2 + 4x - 7)$, $\log (3x^2 + 4x - 7)$, $\log_{10} (3x^2 + 4x - 7)$. Find the values of these derivatives when $x = 3$.

Ex. 2. Find the values of the derivatives of $\log \sqrt{x^3 + 10}$, $\log_{10} \sqrt{x^3 + 10}$, when $x = 2$.

Ex. 3. Differentiate the following: $\log \dfrac{1-x}{1+x}$, $\log \sqrt{\dfrac{1+x}{1-x}}$, $\log \dfrac{1+\sqrt{x}}{1-\sqrt{x}}$, $\log (x + \sqrt{x^2 + a^2})$, $\log(\log x)$, $x \log x$.

Ex. 4. Find anti-derivatives of $\dfrac{2x+3}{x^2+3x+5}$, $\dfrac{3x^2-7}{x^3-7x-1}$, $\dfrac{1}{2x}$.

(*a*) **Logarithmic differentiation.** If

$$y = uvw, \tag{1}$$

then
$$\log y = \log u + \log v + \log w.$$

On differentiation,
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx},$$

whence
$$\frac{dy}{dx} = uvw\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]. \tag{2}$$

This result can easily be reduced to the form obtained in Art. 32. The same method can be used in the case of any finite number of factors. This method of obtaining result (2) is called

the method of logarithmic differentiation. It is frequently more expeditious than that given in Arts. 32, 33, especially when several factors are involved.

Ex. 5. Find $\frac{dy}{dx}$ when $y = \frac{x(x^2+7)^{\frac{1}{2}}}{(x^2+2)^{\frac{1}{3}}}$. (See Ex. 11, Art. 37.)

Here, $\log y = \log x + \frac{1}{2}\log(x^2+7) - \frac{1}{3}\log(x^2+2)$.

On differentiation, $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{x}{x^2+7} - \frac{2x}{3(x^2+2)}$.

From this, on transposing, combining, and reducing,

$$\frac{dy}{dx} = \frac{4x^4+19x^2+42}{3(x^2+7)^{\frac{1}{2}}(x^2+2)^{\frac{4}{3}}}.$$

Ex. 6. Differentiate, with respect to x, the following functions:

(*a*) $\frac{(x+2)^{\frac{3}{2}}}{(4x-7)^{\frac{4}{3}}(3x+5)^{\frac{4}{5}}}$; (*b*) $\frac{(x-1)(x-2)}{(x+1)(x+2)}$; (*c*) $\frac{\sqrt{2x+5}\,\sqrt[3]{7x-5}}{\sqrt[5]{(x+3)^2}}$.

(*b*) Differentiation of an incommensurable (constant) power of a function. This paragraph is supplementary to Art. 37 (*d*).

Let $y = u^n$,

in which n is any constant, commensurable or incommensurable.

Then $\log y = n \log u$.

From this $\frac{1}{y}\frac{dy}{dx} = \frac{n}{u}\frac{du}{dx}$;

and hence $\frac{dy}{dx} = \frac{ny}{u}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}$.

40. Differentiation of a^u.

Put $y = a^u$.

Then $\log y = u \log a$.

On differentiation, $\frac{1}{y}\frac{dy}{dx} = \log a \cdot \frac{du}{dx}$.

$$\therefore \frac{dy}{dx} = y \log a \cdot \frac{du}{dx};$$

i.e. $\frac{d}{dx}(a^u) = a^u \cdot \log a \cdot \frac{du}{dx}$.

If $u = x$, then $\dfrac{d}{dx}(a^x) = a^x \cdot \log a.$

If $a = e$, then $\dfrac{d}{dx}(e^u) = e^u \dfrac{du}{dx}.$

If $u = x$, and $a = e$, then

$$\frac{d}{dx}(e^x) = e^x;$$

that is, the derivative of e^x is itself e^x.

NOTE 1. *On the derivation of results in Arts.* 39, 40. The derivative of $\log_a u$ was deduced by the general and fundamental method, and has been used in finding the derivative of a^u. The latter derivative can be found, however, by the fundamental method, independently of the derivative of $\log_a u$. Moreover, the derivative of $\log_a u$ can be obtained by means of the derivative of a^u. These various methods of finding the derivative of a^u and $\log_a u$ are all employed by writers on the calculus. For examples see Todhunter, *Diff. Cal.*, Arts. 49, 50; Gibson, *Calculus*, § 65, where both these derivatives are obtained independently of each other; Williamson, *Diff. Cal.*, Arts. 29, 30; McMahon and Snyder, *Diff. Cal.*, Arts. 30, 31, where the derivative of the logarithmic function is first obtained and the derivative of the exponential function is deduced therefrom; and Lamb, *Calculus*, Arts. 35 (Ex. 5), 42, where the derivative of the exponential function is obtained first and the derivative of the logarithmic function is deduced therefrom. (See also Echols, *Calculus*, Art. 33 and foot-note.)

NOTE 2. *On the expansion of e^x in a series* see Hall and Knight, *Higher Algebra*, Art. 220; Chrystal, *Algebra*, Vol. II., Chap. XXVIII., §§ 4, 5; and other texts. (This expansion is derived by the calculus in Art. 178, Ex. 7.)

Ex. Assuming the expansion for e^x, show that the derivative of e^x is itself e^x.

NOTE 3. **The compound interest law.** The function e^x "is the only [mathematical] function known to us whose rate of increase is proportional to itself; but there are a great many phenomena in nature which have this property. Lord Kelvin's way of putting it is that 'they follow the compound interest law.'" (See Hall and Knight, *Higher Algebra*, Art. 234, and, in particular, Perry, *Calculus*, Art. 97 and Art. 98, Exs. 4, 2.)

Ex. 1. Differentiate, with respect to x, e^{x^2}, 10^x, 10^{3x^2}, $e^{\sqrt{x}}$.

Ex. 2. Find the t-derivatives of e^{2t}, 10^{t^2}, e^{t^2+3}, 10^{2t+7}.

Ex. 3. Find the x-derivatives of the following:

$$e^x x^m,\quad a^{x^n},\quad \frac{x}{e^x-1},\quad xe^{-x},\quad \frac{e^x-e^{-x}}{e^x+e^{-x}},\quad \frac{e^{x^2}}{x}.$$

Ex. 4. Find anti-derivatives of e^{3x}, xe^{x^2}, $2\,e^{3x+1}$.

41. Differentiation of u^v, in which u and v are both functions of x.

Put $$y = u^v. \tag{1}$$

Then $$\log y = v \log u.$$

On differentiation, $$\frac{1}{y}\frac{dy}{dx} = \frac{v}{u}\frac{du}{dx} + \log u \cdot \frac{dv}{dx}.$$

$$\therefore \frac{dy}{dx} = y\left(\frac{v}{u}\frac{du}{dx} + \log u \cdot \frac{dv}{dx}\right);$$

i.e. $$\frac{d}{dx}(u^v) = u^v\left(\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right). \tag{2}$$

NOTE 1. It is better not to memorize result (2), but merely to note the fact that the function in (1) is easily treated by the method of logarithmic differentiation.

Note 2. The beginner needs to guard against confusing the derivatives of the functions u^n, a^u, and u^v.

Ex. 1. Find $\frac{dy}{dx}$ when $$y = x^x.$$

Here $$\log y = x \log x.$$

On differentiation, $$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \log x;$$

whence $$\frac{dy}{dx} = x^x(1 + \log x).$$

Ex. 2. Find the x-derivatives of

$$(3x+7)^{x^2},\ (3x+7)^{2x},\ \{(3x+7)^x\}^2,\ \sqrt[x]{x},\ x^{x^n},\ e^{e^x},\ \left(\frac{e}{x}\right)^{\frac{x}{e}},\ \log\frac{x}{a^x}.$$

C. Trigonometric Functions.

42. Differentiation of $\sin u$.

Put $$y = \sin u.$$

Then $$y + \Delta y = \sin(u + \Delta u).$$

$$\therefore \Delta y = \sin(u + \Delta u) - \sin u$$

$$= 2\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2}. \quad \text{(Trigonometry)}$$

$$\therefore\ \frac{\Delta y}{\Delta x} = 2\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2}\cdot\frac{1}{\Delta x}$$

$$= \cos\left(u + \frac{\Delta u}{2}\right)\cdot\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}\cdot\frac{\Delta u}{\Delta x}.$$

Let $\Delta x \doteq 0$; then also $\Delta u \doteq 0$, and

$$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta u \doteq 0}\cos\left(u + \frac{\Delta u}{2}\right)\cdot\lim_{\Delta u \doteq 0}\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}\cdot\lim_{\Delta x \doteq 0}\frac{\Delta u}{\Delta x};$$

i.e.
$$\frac{dy}{dx} = \cos u\cdot 1\cdot\frac{du}{dx};$$

i.e.
$$\frac{d}{dx}(\sin u) = \cos u\,\frac{du}{dx}. \qquad (1)$$

In particular, if $u = x$,

$$\frac{d}{dx}(\sin x) = \cos x. \qquad (2)$$

That is, the rate of change of the sine of an angle with respect to the angle is equal to the cosine of the angle.

NOTE 1. Result (2) can also be obtained by geometry. (Ex. *Show this.*) See Williamson, *Diff. Cal.*, Art. 28, and other texts.

NOTE 2. Result (2) shows that as the angle x increases from 0 to $\frac{\pi}{2}$ *the rate of increase* of the sine is positive, since $\cos x$ is then positive. As x increases from $\frac{\pi}{2}$ to π the rate is negative (*i.e.* the sine decreases), since $\cos x$ is then negative. The rate is negative when x increases from π to $\frac{3\pi}{2}$, and the rate is positive when x increases from $\frac{3\pi}{2}$ to 2π. This agrees with what is shown in elementary trigonometry, and it is also apparent on a glance at the curve $y = \sin x$.

NOTE 3. Result (2) also shows that if the angle increases at a uniform rate, the sine increases the faster the nearer the angle is to zero, and increases more slowly as the angle approaches 90°. This is also apparent from an inspection of a table of natural sines, or from a glance at the curve $y = \sin x$.

Note 4. The derivative of $\sin u$ has been found by the general and fundamental method of differentiation. It is not necessary to use this

method in finding the derivatives of the remaining trigonometric and anti-trigonometric functions, for these derivatives can be deduced from that of the sine.

Ex. 1. Find the x-derivatives of $\sin 2u$, $\sin 3u$, $\sin \frac{1}{2}u$, $\sin \frac{3}{4}u$, $\sin \frac{17}{3}u$.

Ex. 2. Find the x-derivatives of $\sin 2x$, $\sin 3x$, $\sin \frac{1}{2}x$, $\sin 3x^2$, $\sin^2 3x$, $\sin 4x^5$, $\sin^5 4x$.

Ex. 3. Find the derivatives with respect to t of $\sin 5t$, $\sin \frac{1}{2}t^2$.

Ex. 4. Find the x-derivatives of $\dfrac{\sin 2x}{\sin 3x}$, $x \sin 2x$, $x^2 \sin\left(x + \dfrac{\pi}{4}\right)$.

Ex. 5. At what angles does the curve $y = \sin x$ cross the x-axis?

Ex. 6. At what points on the curve $y = \sin x$ is the tangent inclined 30° to the x-axis.

Ex. 7. Draw the curve $y = \sin 2x$. At what angles does it cross the x-axis?

Ex. 8. Draw the curve $y = \sin x + \cos x$. Where does it cross the x-axis? At what angles does it cross the x-axis? Where is it parallel to the x-axis?

Ex. 9. Find the x-derivatives of the following: $\sin nx$, $\sin x^n$, $\sin^n x$, $\sin(1 + x^2)$, $\sin(nx + a)$, $\sin(a + bx^n)$, $\sin^3 4x$, $\dfrac{\sin x}{x}$, $\sin(\log x)$, $\log(\sin x)$, $\sin(e^x) \cdot \log x$.

Ex. 10. (*a*) Find anti-derivatives of

$$\cos x,\ \cos 3x,\ \cos(2x + 5),\ x \cos(x^2 - 1).$$

(*b*) Find anti-differentials of $\cos 2x\, dx$, $\cos(3x - 7)dx$, $x^2 \cos x^3\, dx$.

Ex. 11. Calculate $d(\sin x)$ when $x = 46°$ and $dx = 20'$, and compare the result with $\sin 46° 20' - \sin 46°$. (Radian measure must be used in the computation.)

Ex. 12. Compare $d(\sin x)$ when $x = 20°$ and $dx = 30'$, with

$$\sin 20° 30' - \sin 20°.$$

43. Differentiation of $\cos u$.

Put $\qquad y = \cos u.$

Then $\qquad y = \sin\left(\dfrac{\pi}{2} - u\right).$

$$\therefore \frac{dy}{dx} = \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right) \qquad \text{[Art. 42, Eq. (1)]}$$

$$= -\sin u \frac{du}{dx};$$

i.e.
$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}. \tag{1}$$

In particular, if $u = x$,

$$\frac{d}{dx}(\cos x) = -\sin x. \tag{2}$$

Ex. 1. Obtain derivative (1) by the fundamental method.

Ex. 2. Show that result (2) agrees in a general way with what is shown in trigonometry about the behaviour of the cosine as the angle changes from 0° to 360°. Also inspect the curve $y = \cos x$.

Ex. 3. Find where the curve $y = \cos x$ is parallel to the x-axis, and where its slope is $\tan 25°$.

Ex. 4. Show that the tangents of the curve $y = \cos x$ cannot cross the x-axis at an angle between $+45°$ and $+135°$.

Ex. 5. Find the slope of the tangent to the ellipse $x = a\cos\theta$, $y = b\sin\theta$. (See Art. 35.)

Ex. 6. Find the slope of the tangent to the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$. What angle does this tangent make with the x-axis when $a = 5$, and $\theta = \frac{\pi}{3}$?

Ex. 7. Find the x derivatives of the following: $\cos(2x+5)$, $\cos^3 5x$, $x^2\cos x$, $\frac{1-\cos x}{1+\cos x}$, $\cos mx \cos nx$, $xe^{\cos x}$, $e^{ax}\cos mx$.

Ex. 8. Find anti-differentials of $\sin x\,dx$, $\sin\frac{1}{2}x\,dx$, $\sin(3x-2)dx$, $x\sin(x^2+4)dx$.

Ex. 9. Calculate $d\cos x$ when $x = 57°$ and $dx = 30'$, and compare the result with $\cos 57°\,30' - \cos 57°$.

44. Differentiation of tan u.

Put $$y = \tan u.$$

Then $$y = \frac{\sin u}{\cos u}.$$

$$\therefore \frac{dy}{dx} = \frac{\cos u\frac{d}{dx}(\sin u) - \sin u\frac{d}{dx}(\cos u)}{\cos^2 u}$$

$$= \frac{(\cos^2 u + \sin^2 u)}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{1}{\cos^2 u}\frac{du}{dx} = \sec^2 u\frac{du}{dx};$$

i.e. $$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}. \tag{1}$$

If $u = x$, then $$\frac{d}{dx}(\tan x) = \sec^2 x. \tag{2}$$

Ex. 1. Show the agreement of result (2) with the facts of elementary trigonometry, and with the curve $y = \tan x$.

Ex. 2. Show that the tangents of the curve $y = \tan x$ cross the x-axis at angles varying from $+45°$ to $+90°$.

Ex. 3. State the x-derivatives of $\tan 2u$, $\tan 3u$, $\tan mu$, $\tan nu^2$, $\tan 2x$, $\tan \frac{1}{2}x$, $\tan mx$, $\tan 3x^2$, $\tan 4x^3$, $\tan mx^n$, $\tan^2 3x$, $\tan^3 4x$, $\tan^n mx$, $\tan^2(\frac{2}{3}x + 3)$, $\log \tan \frac{x}{2}$.

Ex. 4. Find anti-differentials of $\sec^2 x dx$, $\sec^2 2x\,dx$, $\sec^2(3x + a)dx$.

Ex. 5. Compute $d \tan x$ when $x = 20°$, $dx = 20'$, and compare the result with $\tan 20° 20' - \tan 20°$.

Ex. 6. When is the differential of $\tan x$ infinitely great?

45. Differentiation of cot u.

Either, substitute $\frac{\cos u}{\sin u}$, for $\cot u$, and proceed as in Art. 44;

or, substitute $\tan(90° - u)$ for $\cot u$, and proceed as in Art. 43;

or, substitute $\frac{1}{\tan u}$ for $\cot u$, and differentiate. It will be found that

$$\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u \frac{du}{dx}. \tag{1}$$

If $u = x$, $$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x. \tag{2}$$

Ex. Show the general agreement of result (2) with the facts of elementary trigonometry, and with the curve $y = \cot x$.

46. Differentiation of sec u.

Put $$y = \sec u = \frac{1}{\cos u}.$$

Then $$\frac{dy}{dx} = \frac{\sin u}{\cos^2 u} \cdot \frac{du}{dx} = \frac{1}{\cos u} \cdot \frac{\sin u}{\cos u} \cdot \frac{du}{dx};$$

i.e. $$\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}. \tag{1}$$

If $u = x$, $$\frac{d}{dx}(\sec x) = \sec x \tan x. \tag{2}$$

47. Differentiation of csc u.

Put $$y = \csc u = \frac{1}{\sin u}. \quad \text{Then } \frac{dy}{dx} = -\frac{\cos u}{\sin^2 u}\frac{du}{dx}.$$

That is, $$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}. \tag{1}$$

If $u = x$, $$\frac{d}{dx}(\csc x) = -\csc x \cot x. \tag{2}$$

NOTE. Or put $y = \csc u = \sec\left(\frac{\pi}{2} - u\right)$, and proceed as in Art. 43.

48. Differentiation of vers u. Put $y = \text{vers } u = 1 - \cos u$. Then, on differentiation,
$$\frac{d}{dx}(\text{vers } u) = \sin u \frac{du}{dx}.$$

In particular, if $u = x$,
$$\frac{d}{dx}(\text{vers } x) = \sin x.$$

Ex. 1. Find the x-derivatives of $\cot(2x + 3)$, $\sec(\frac{1}{2}x + 3)$, $\csc(3x - 7)$, $\text{vers}(5x + 2)$, $\sec^n x$.

Ex. 2. Find the t-derivatives of $\cot^2(3t + 1)$, $\sec^3(\frac{1}{3}t - 1)$, $\csc^2 \frac{3}{5}(t + 5)$, $\cot(9t^2)$, $\sec(7t - 2)^2$.

Ex. 3. Show that $D \log(\tan x + \sec x) = D \log \tan(\frac{1}{4}\pi + \frac{1}{2}x) = \sec x$.

D. INVERSE TRIGONOMETRIC FUNCTIONS.*

49. Differentiation of $\sin^{-1} u$.

Put $$y = \sin^{-1} u.$$

Then $$\sin y = u.$$

On differentiation, $$\cos y \frac{dy}{dx} = \frac{du}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}\frac{du}{dx};$$

i.e. $$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}. \tag{1}$$

If $u = x$, $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}. \tag{2}$$

* See Murray, *Plane Trigonometry*, Arts. 17, 88.

NOTE 1. **On the ambiguity of the derivative of $\sin^{-1} x$.** The result in (2) is ambiguous, since the sign of the radical may be positive or negative. This ambiguity is apparent on looking at the curve $y = \sin^{-1} x$, Fig. 14.

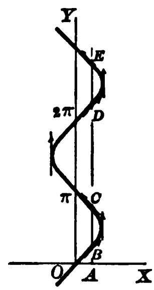

FIG. 14.

Draw the ordinate $ABCDE$ at $x = x_1$. The tangents at B and D make acute angles with the x-axis, and the tangents at C and E make obtuse angles with the x-axis. Hence, at B and D $\frac{dy}{dx}$ is positive; and at C and E $\frac{dy}{dx}$ is negative. That is, at B and D $\frac{d}{dx}(\sin^{-1} x) = \frac{+1}{\sqrt{1 - x_1^2}}$; and at C and E $\frac{d}{dx}(\sin^{-1} x) = \frac{-1}{\sqrt{1 - x_1^2}}$. Thus the sign of $\frac{d}{dx}(\sin^{-1} x)$ depends upon the particular value taken of the infinite number of values of y which satisfy the equation $y = \sin^{-1} x$.

NOTE 2. If it is understood that there be taken the *least* positive value of y satisfying the equation $y = \sin^{-1} x_1$ (in which x_1 is positive), then the sign of the derivative is positive. Similar considerations are necessary in (1).

Ex. 1. Show by the graph in Fig. 14, or otherwise, that when $x = 1$, $\frac{d}{dx}(\sin^{-1} x) = +\infty$, and that when $x = -1$, $\frac{d}{dx}(\sin^{-1} x)$ is $-\infty$.

Ex. 2. Find the x-derivatives of

$$\sin^{-1} x^n,\quad \sin^{-1}\frac{x+1}{\sqrt{2}},\quad \sin^{-1}\frac{2x}{1+x^2},\quad \sin^{-1}\frac{2x}{\sqrt{1-x^2}},$$

$$\sin^{-1}\sqrt{1-x^2},\quad \sqrt{1-x^2}\cdot\sin^{-1} x - x,\quad \sin^{-1}\sqrt{\sin x}.$$

Ex. 3. Show that a tangent to the curve $y = \sin^{-1} x$ cannot cross the x-axis at an angle *between* $-45°$ and $+45°$.

Ex. 4. Find anti-derivatives of $\frac{1}{\sqrt{1-x^2}}$, $\frac{2x}{\sqrt{1-x^4}}$, $\frac{x^2}{\sqrt{1-x^6}}$.

50. Differentiation of $\cos^{-1} u$.

Put $$y = \cos^{-1} u.$$

Then $$\cos y = u.$$

On differentiation, $$-\sin y\frac{dy}{dx} = \frac{du}{dx}.$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sin y}\frac{du}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}\frac{du}{dx};$$

i.e. $$\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

If $u = x$, $$\frac{d}{dx}(\cos^{-1}) = -\frac{1}{\sqrt{1-x^2}}.$$

Ex. 1. Explain the ambiguity of sign in the derivative of $\cos^{-1} x$ by means of the curve $y = \cos^{-1} x$. Show that if there be taken the least positive value of y satisfying $y = \cos^{-1} x$, in which x is positive, the sign of the derivative is negative.

Ex. 2. Determine the angles at which the tangents touching the curve $y = \cos^{-1} x$ where $x = \frac{1}{\sqrt{2}}$, cross the x-axis.

Ex. 3. Find the x-derivatives of $\cos^{-1}\frac{x^{2n}-1}{x^{2n}+1}$, $\cos^{-1}\frac{1-x^2}{1+x^2}$, $a\cos^{-1}\frac{a-x}{a}$.

51. Differentiation of $\tan^{-1} u$.

Put $$y = \tan^{-1} u.$$

Then $$\tan y = u.$$

On differentiation, $$\sec^2 y\frac{dy}{dx} = \frac{du}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y}\frac{du}{dx} = \frac{1}{1+\tan^2 y}\frac{du}{dx};$$

i.e. $$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

In particular, if $u = x$,

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

NOTE. The derivative of $\tan^{-1} x$ is always positive. This is also evident on a glance at the curve $y = \tan^{-1} x$.

Ex. 1. Find the x-derivatives of $\tan^{-1} 2x$, $\tan^{-1} 2y$, $\tan^{-1} x^2$, $\tan^{-1} y^3$.

Ex. 2. Find the t-derivatives of $\tan^{-1} 4t$, $\tan^{-1} t^4$, $\tan^{-1} 3x^2$.

Ex. 3. Show that the angles made with the x-axis by the tangents to the curve $y = \tan^{-1} x$ are 0°, 45°, and the angles between 0° and 45°.

Ex. 4. Show how to determine the abscissas of the points of $y = \tan^{-1} x$, the tangents at which cross the x-axis at an angle of 30°.

Ex. 5. Find the x-derivatives of the following: $\tan^{-1}\frac{2x}{1-x^2}$, $\tan^{-1}\frac{x}{1+x^2}$, $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$, $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $\tan^{-1}\sqrt{\frac{x}{a+x}}$, $\tan^{-1}\frac{3a^2x-x^3}{a(a^2-3x^2)}$.

Ex. 6. (*a*) Show that $D \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}} = \dfrac{1}{2}$. (*b*) Show, by differentiation, that $D\left(\tan^{-1} x + \tan^{-1}\dfrac{1}{x}\right)$ is independent of x.

Ex. 7. Find anti-differentials of $\dfrac{dx}{1+x^2}$, $\dfrac{2\,x\,dx}{1+x^4}$, $\dfrac{x^3\,dx}{1+x^8}$.

52. Differentiation of $\cot^{-1} u$. On proceeding in a manner similar to that in Art. 51, it will be found that

$$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2}\frac{du}{dx}.$$

If $u = x$,
$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}.$$

Ex. 1. Show, by means of the curve $y = \cot^{-1} x$, that the derivative of $\cot^{-1} x$ is always negative.

Ex. 2. Find the x-derivative of $\cot^{-1}\dfrac{a}{x} + \log\sqrt{\dfrac{x-a}{x+a}}$.

53. Differentiation of $\sec^{-1} u$.

Put
$$y = \sec^{-1} u.$$

Then
$$\sec y = u.$$

On differentiation, $\sec y \tan y \dfrac{dy}{dx} = \dfrac{du}{dx}$.

$$\therefore \frac{dy}{dx} = \frac{1}{\sec y \tan y}\frac{du}{dx} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}}\frac{du}{dx};$$

i.e.
$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}. \tag{1}$$

If $u = x$, then
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}. \tag{2}$$

Ex. 1. Explain the ambiguity of the result (2). Show that, when x is positive, the positive value of the radical is taken with the least positive value of $\sec^{-1} x$.

Ex. 2. Find the x-derivatives of $\sec^{-1} x^2$, $\sec^{-1}\dfrac{1}{2\,x^2-1}$, $\sec^{-1}\dfrac{a}{\sqrt{a^2-x^2}}$, $\sec^{-1}\dfrac{x^2+1}{x^2-1}$.

Ex. 3. Show by differentiation that $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}} - \sec^{-1}\dfrac{1}{\sqrt{1-x^2}}$ is independent of x.

54. Differentiation of cosec^{-1} u. On proceeding in a manner similar to that in Art. 53, it will be found that

$$\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}. \tag{1}$$

If $u = x$, $$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}. \tag{2}$$

Ex. 1. Explain the ambiguity in sign in (2) by means of the graph of $\csc^{-1} u$. Show that, when x is positive, the negative value of the radical is taken with the least positive value of $\csc^{-1} u$.

55. Differentiation of vers^{-1} u.

Put $$y = \text{vers}^{-1} u.$$

Then $$\text{vers}\, y = u.$$

On differentiation, $$\sin y \frac{dy}{dx} = \frac{du}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sin y}\frac{du}{dx} = \frac{1}{\sqrt{1-\cos^2 y}}\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-(1-\text{vers}\, y)^2}}\frac{du}{dx};$$

i.e. $$\frac{d}{dx}(\text{vers}^{-1} u) = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx}. \tag{1}$$

If $u = x$, $$\frac{d}{dx}(\text{vers}^{-1} x) = \frac{1}{\sqrt{2x-x^2}}. \tag{2}$$

Ex. 1. Find the x-derivative of $\text{vers}^{-1}\frac{2x^2}{1+x^2}$.

56. Differentiation of implicit functions: two variables.

N.B. Examples of the differentiation of implicit functions have been given in Exs. 13, 14, Art. 37. A preliminary study of these examples will help to make this article clear.

Let y be an implicit function of x, the function y and the variable x being connected by a relation

$$f(x, y) = c. \tag{1}$$

If, as sometimes happens, it is impossible or inconvenient to express y as an explicit function of x, the derivative $\frac{dy}{dx}$ may be obtained in the following way:

On taking the x-derivative of each member of (1), there is obtained a result of the form

$$P + Q\frac{dy}{dx} = 0. \tag{2}$$

From this
$$\frac{dy}{dx} = -\frac{P}{Q}. \tag{3}$$

Since the x-derivative of $f(x, y)$ is $P + Q\frac{dy}{dx}$, *the differential of* $f(x, y)$ *is* (Art. 27) $Pdx + Q\frac{dy}{dx}dx$, *i.e.* (Art. 27) $Pdx + Qdy$.

Ex. 1. Find $\frac{dy}{dx}$, when $xy = c$.

Differentiation of the members of this equation gives $y + x\frac{dy}{dx} = 0$; whence $\frac{dy}{dx} = -\frac{y}{x}$. The x-derivative of xy is $y + x\frac{dy}{dx}$; accordingly, the differential of xy is $x\,dy + y\,dx$. [Compare result (7), Art. 32.]

Ex. 2. Write the differentials of the first members of the equations in Exs. 13, 14, Art. 37.

Ex. 3. Find $\frac{dy}{dx}$ in each of the following cases: (i) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$; (ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; (iii) $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$; (iv) $(\cos x)^y - (\sin y)^x = 0$.

Ex. 4. Write the differentials of the first members of the equations in Ex. 3.

Note 1. It should be observed, as illustrated in Equation (2) and the above examples, that when the differential of $f(x, y)$ is written $Pdx + Qdy$, P is the same expression as is obtained by differentiating $f(x, y)$ with respect to x, and at the same time regarding y as constant or letting y remain constant, and Q is the same expression as is obtained by differentiating $f(x, y)$ with respect to y, and at the same time regarding x as constant or letting x remain constant. Here P is called *the partial x-derivative* of $f(x, y)$, and Q is called *the partial y-derivative* of $f(x, y)$. These *partial* derivatives are denoted by the symbols $\frac{\partial f(x, y)}{\partial x}$ and $\frac{\partial f(x, y)}{\partial y}$ respectively. With this notation, result (3) may be written

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f(x, y)}{\partial x}}{\dfrac{\partial f(x, y)}{\partial y}}, \quad \text{or} \quad -\frac{\dfrac{\partial}{\partial x}f(x, y)}{\dfrac{\partial}{\partial y}f(x, y)}. \tag{4}$$

Ex. 5. In the exercises above, test the first statement made in this note.

NOTE 2. Partial derivatives and the differentiation of implicit functions are discussed further in Chapter VIII.

EXAMPLES.

N.B. It is not advisable for the beginner to work the larger part of Exs. 1–8 before proceeding to the next chapter. Many of the differentiations required in these examples are far more difficult than those that are commonly met in pure and applied mathematics; but the exercise in working a fair proportion of them will develop a skill and confidence that will be a great aid in future work.

Differentiate the functions in Exs. 1–4, 6, 7, with respect to x.

1. (i) $(2x-1)(3x+4)(x^2+11)$; (ii) $(a+x)(b+x)$;
(iii) $(a+x)^m(b+x)^n$; (iv) $\dfrac{(x+a)^m}{(x+b)^n}$; (v) $\dfrac{x^m}{(1+x)^n}$; (vi) $\dfrac{x}{\sqrt{a^2-x^2}}$;
(vii) $\dfrac{x}{\sqrt{1+x^2}}$; (viii) $\dfrac{\sqrt{a+x}}{\sqrt{a}+\sqrt{x}}$; (ix) $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$;
(x) $\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)^n$; (xi) $x(a^2+x^2)\sqrt{a^2-x^2}$.

2. The logarithms of: (i) $7x^4+3x^2-17x+2$; (ii) $\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$;
(iii) $\dfrac{x}{a-\sqrt{a^2-x^2}}$; (iv) $\sqrt{\dfrac{1+\sin x}{1-\sin x}}$; (v) $\sqrt{\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}$.

3. (i) $\sin 4x^5$; (ii) $\cos^2 7x$; (iii) $\sec^2 3x$; (iv) $\tan(8x+5)$;
(v) $x^m \log x$; (vi) $\sin^p x^q$; (vii) $\sin nx \cdot \sin^n x$; (viii) $\sin(\sin x)$;
(ix) $\sin(\log nx)$; (x) $\log(\sin nx)$.

4. (i) $\log \sqrt[4]{\dfrac{x-1}{x+1}} - \dfrac{1}{2}\tan^{-1}x$; (ii) $\log\sqrt{\dfrac{\tan x-1}{\tan x+1}} - x$;
(iii) $\log\sqrt[4]{\dfrac{1+x}{1-x}} - \dfrac{1}{2}\tan^{-1}x$.

5. Show that $D\left\{\dfrac{x\sqrt{a^2+x^2}}{2} + \dfrac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)\right\} = \sqrt{a^2+x^2}$.

6. (i) $\tan^{-1} e^x$; (ii) $\sin^{-1}(\cos x)$; (iii) $\sin(\cos^{-1}x)$;
(iv) $\tan^{-1}(n\tan x)$; (v) $\sin^{-1}\dfrac{b+a\cos x}{a+b\cos x}$; (vi) $e^{ax}\sin^m rx$;
(vii) $\tan a^{\frac{1}{x}}$; (viii) $e^x\sqrt{\dfrac{1+x}{1-x}}$.

7. (i) $\left(\frac{x}{n}\right)^{nx}$; (ii) $\frac{c^2}{x}e^{\frac{x}{c}}$; (iii) x^{e^x}; (iv) e^{x^x}; (v) $x^{(x^x)}$; (vi) $(x^x)^x$.

8. Find $\frac{dy}{dx}$ under each of the following conditions:
(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$; (ii) $(x^2 + y^2)^2 - a^2(x^2 - y^2) = 0$; (iii) $x^2y^4 + \sin y = 0$; (iv) $\sin(xy) = mx$; (v) $\sin x \sin y + \sin x \cos y = y$; (vi) $e^y - e^x + xy = 0$; (vii) $x^y = y^x$; (viii) $ye^{ny} = ax^m$.

9. Find $\frac{dy}{dx}$ in terms of x, when $x = e^{\frac{x-y}{y}}$.

10. Differentiate as follows: (i) $3y^2 - 7y + 11$ with respect to $3y$; (ii) $4t^2 - 11t + 1$ with respect to $t + 2$; (iii) x with respect to $\sin x$; (iv) $\sin z$ with respect to $\cos z$; (v) x with respect to $\sqrt{1 - x^2}$.

11. (i) Given $y = 3u^2 - 7u + 2$ and $u = 2x^3 + 3x + 2$, find $\frac{dy}{dx}$; (ii) given $y = e^s + s^2$ and $s = \tan t$, find $\frac{dy}{dt}$; (iii) given $v = \sqrt{2gs}$, $s = \frac{1}{2}gt^2$, find $\frac{dv}{dt}$ in two ways; (iv) $u = \tan^{-1}(xy)$, $y = e^x$, find $\frac{du}{dx}$.

12. Compute the angle at which the following curves intersect, and sketch the curves: (i) $x^2 - y^2 = 9$ and $xy = 4$; (ii) $x^2 + y^2 = 25$ and $4y^2 = 9x$; (iii) $y^2 = 8(x + 2)$ and $y^2 + 4(x - 1) = 0$; (iv) $y = 3x^2 - 1$ and $y = 2x^2 + 3$; (v) $x^2 + y^2 = 9$ and $(x - 4)^2 + y^2 - 2y = 15$.

13. A point P is moving with uniform speed along a circle of radius a and centre O; AB is any diameter, and Q is the foot of the perpendicular from P on AB. Show that the speed of Q is variable, that at A and B it is zero, and at O it is equal to the speed of P. (The motion of Q is called *simple harmonic motion.*)

[SUGGESTION: Denote angle AOP by θ, and OQ by x. Then $x = a\cos\theta$; hence $\frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt}$.]

14. Suppose, in Ex. 13, the radius is 18 inches, and P is making 4 revolutions per second: what is the speed of Q when AOP is 15°, 30°, 45°, 60°, 75°, 90°, 120°, 150°, respectively?

CHAPTER V.

SOME GEOMETRICAL, PHYSICAL, AND ANALYTICAL APPLICATIONS. GEOMETRIC DERIVATIVES AND DIFFERENTIALS.

N.B. *The variation of functions, the sketching of graphs, and the determination of* **maxima and minima,** *which are discussed in Chapter VII., can be studied before entering upon this chapter.* For some reasons it may be preferable to do this.

57. This chapter gives some practical applications of the preceding principles of the calculus. The applications in Arts. 58 and 59 are already familiar or obvious. Rolle's theorem and the theorem of mean value, in Arts. 63, 64, are two of the most important theorems in the calculus. The study of the geometric derivatives and differentials, in Art. 67, is of no immediate pressing importance, but will be found of particular interest when Chapters XII. and XVI. are taken up. A glance over this article, however, will serve to make clearer and stronger the notions of a derivative and a differential.

58. Slope of a curve at any point: rectangular coördinates. It has been shown in Art. 24 that at a point (x_1, y_1) on a curve whose equation is (1) $y = f(x)$, or (2) $\phi(x, y) = 0$, the slope of the tangent is $\frac{dy_1}{dx_1}$. [Here $\frac{dy_1}{dx_1}$ denotes the result of substituting (x_1, y_1) for (x, y) in $\frac{dy}{dx}$ derived from (1) or (2).] Examples have been given in the preceding articles.

59. Lengths of tangent, subtangent, normal, and subnormal, for any point on a curve: rectangular coördinates. Let P be a point (x_1, y_1) on the curve $y = f(x)$ [or, $\phi(x, y) = 0$].

At P let the tangent PT be drawn; likewise the normal PN and the ordinate PM. The length of the line PT, namely, that

part of the tangent which is intercepted between P and the x-axis, is here termed *the length of the tangent.* The projection of TP on the x-axis, namely TM, is called the *subtangent.* The length of the line PN, the *part* of the normal which is intercepted between P and the x-axis, is termed *the length of the normal.* The projection of PN on the x-axis, namely MN, is called the *subnormal.*

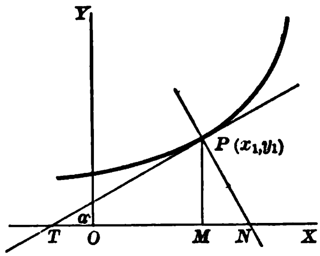

FIG. 15.

NOTE 1. The subtangent is measured from the intersection of the tangent with the x-axis *to* the foot of the ordinate; the subnormal is measured *from* the foot of the ordinate to the intersection of the normal with the x-axis.

Let angle XTP be denoted by α; then $\tan\alpha = \frac{dy_1}{dx_1}$. In the triangle TPM: $MP = y_1$; $TM = y_1 \cot\alpha = y_1 \frac{dx_1}{dy_1}$; $TP = y_1 \csc\alpha$ $= y_1\sqrt{1+\left(\frac{dx_1}{dy_1}\right)^2}$; $\left(\text{or, } TP = \sqrt{\overline{MP}^2 + \overline{TM}^2} = y_1\sqrt{1+\left(\frac{dx_1}{dy_1}\right)^2}\right)$. In the triangle PMN: angle $MPN = \alpha$; $MN = y_1 \tan MPN = y_1\frac{dy_1}{dx_1}$; $PN = y_1 \sec MPN = y_1\sqrt{1+\left(\frac{dy_1}{dx_1}\right)^2}$; $\left(\text{or, } PN = \sqrt{\overline{MP}^2 + \overline{MN}^2} = y_1\sqrt{1+\left(\frac{dy_1}{dx_1}\right)^2}\right)$.

It being understood that y and $\frac{dy}{dx}$ denote the ordinate and the slope of the tangent at any point on the curve, these results may be written:

$$\textbf{subtangent} = y\frac{dx}{dy};$$

$$\textbf{subnormal} = y\frac{dy}{dx};$$

$$\textbf{length of tangent} = y\sqrt{1+\left(\frac{dx}{dy}\right)^2};$$

$$\textbf{length of normal} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

NOTE 2. These results are true, no matter what the figure may be. The student is advised to draw various figures.

NOTE 3. These results may also be derived by means of analytic geometry. For, since $\tan PTX = \frac{dy_1}{dx_1}$,

the equation of the tangent at P is $$y - y_1 = \frac{dy_1}{dx_1}(x - x_1); \qquad (1)$$

and the equation of the normal at P is $$(y - y_1)\frac{dy_1}{dx_1} + (x - x_1) = 0. \qquad (2)$$

Hence, from (1), the intercept $OT = x_1 - y_1\frac{dx_1}{dy_1}$;

and from (2) the intercept $ON = x_1 + y_1\frac{dy_1}{dx_1}$.

The subtangent $TM = OM - OT = y_1\frac{dx_1}{dy_1}$.

The subnormal $MN = ON - OM = y_1\frac{dy_1}{dx_1}$.

Then TP and PN can be found from MP, TM, and MN.

EXAMPLES.

N.B. *Sketch all the curves and draw all the lines involved in the following examples.*

1. In each of the following curves write the equations of the tangent and the normal, and find the lengths of the subnormal, subtangent, tangent, and normal, at any point (x_1, y_1), or at the point more particularly described: (1) Circle $x^2 + y^2 = 25$ where $x = -3$; (2) parabola $y^2 = 8x$ at $x = 2$; (3) ellipse $b^2x^2 + a^2y^2 = a^2b^2$; (4) sinusoid $y = \sin x$; (5) exponential curve $y = e^x$.

2. Where is the curve $y(x - 2)(x - 3) = x - 7$ parallel to the x-axis?

3. What must a^2 be in order that the curves $16x^2 + 25y^2 = 400$ and $49x^2 + a^2y^2 = 441$ intersect at right angles?

4. In the exponential curve $y = be^{\frac{x}{a}}$ show that the subtangent is constant and that the subnormal is $\frac{y^2}{a}$.

5. In the semi-cubical parabola $3y^2 = (x + 1)^3$ show that the subnormal varies as the square of the subtangent.

6. In the hypocycloid of four cusps, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$: (1) Write the equation of the tangent at (x_1, y_1); (2) show that the part of the tangent intercepted between the axes is of constant length a; (3) show that the length of the perpendicular from the origin on the tangent at (x, y) is $\sqrt[3]{axy}$; (4) if p, p_1 be the lengths of the perpendiculars from the origin to the tangent and normal at any point on the curve, $4p^2 + p_1^2 = a^2$.

7. In the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, write the equation of the tangent at any point (x_1, y_1), and show that the sum of the intercepts made on the axes by this tangent is constant. Show that this curve touches the axes at $(a, 0)$ and $(0, a)$.

8. In the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$: (1) Calculate the lengths of the subnormal, subtangent, normal, and tangent at any point (x, y); (2) show that the tangent at any point crosses the y-axis at the angle $\frac{\theta}{2}$; (3) show that the part of the tangent intercepted between the axes is $a\theta \operatorname{cosec} \frac{\theta}{2}$.

9. In the hyperbola $xy = c^2$: (1) Show that for any point (x, y) on the curve the subnormal is $-\frac{y^3}{c^2}$ and the subtangent is $-x$; (2) find the x- and y-intercepts of the tangent at any point (x_1, y_1), and thence deduce a method of drawing the tangent and normal to the curve at any point on it. Show that the product of these intercepts is $4c^2$.

10. In the semi-cubical parabola $ay^2 = x^3$, show that the length of the subtangent for any point (x, y) is $\frac{2}{3}x$; thence deduce a way of drawing the tangent and the normal to the curve at any point on it.

11. Show that the parabola $x^2 = 4y$ intersects the witch $y = \frac{8}{x^2+4}$ at an angle $\tan^{-1} 3$; *i.e.* $71° 33' 54''$.

12. Find at what angles the parabola $y^2 = 2ax$ cuts the folium of Descartes $x^3 + y^3 = 3axy$.

13. In the curve $x^m y^n = a^{m+n}$ show: (1) That the subtangent for any point varies as the abscissa of the point; (2) that the portion of the tangent intercepted between the axes is divided at its point of contact into segments which are to each other in the constant ratio $m : n$; (3) thence, deduce a method of drawing the tangent and the normal at any point on the curve. (The curves $x^m y^n = a^{m+n}$, obtained by giving various values to m and n, are called *adiabatic curves.* Instances of these curves are given in Exs. 9, 10, and in the parabolas in Exs. 11, 12.)

14. Show that all the curves obtained by giving different values to n in $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$, touch one another at the point (a, b). Draw the curves in which (a, b) is $(4, 7)$, $n = 1$, $n = 2$.

15. Show that the tangents at the points where the parabola $ay = x^2$ meets the folium of Descartes $x^3 + y^3 = 3axy$ are parallel to the x-axis, and that the tangents at the points where the parabola $y^2 = ax$ meets the folium are parallel to the y-axis. Make figures for the curves in which $a = 1$ and $a = 4$.

60. Slope of a curve at any point: polar coördinates. Let CM be a curve whose equation is $r=f(\theta)$, [or $\phi(r, \theta)=0$], and P be any point on it having coördinates r_1, θ_1, with reference to the pole O and the initial line OL. Draw OP; then $OP=r_1$, and angle $LOP=\theta_1$. Through P and Q (a neighbouring point on the curve), draw the chord TPQ, and draw OQ. From P draw PR at right angles to OQ.

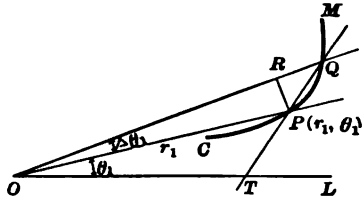

FIG. 16.

Let angle $\quad POQ=\Delta\theta_1$, and $OQ=r_1+\Delta r_1$;

then $\quad PR=r_1 \sin \Delta\theta_1$, and $RQ=r_1+\Delta r_1-r_1 \cos \Delta\theta_1$.

The angle between the radius vector drawn to any point P and the tangent at P is usually denoted by ψ. Since

$$\psi=\lim_{\Delta\theta_1 \doteq 0} \text{angle } RQP,$$

then, using the general coördinates r, θ, instead of r_1, θ_1,

$$\begin{aligned}\tan\psi &= \lim_{\Delta\theta \doteq 0} \frac{RP}{QR} \\ &= \lim_{\Delta\theta \doteq 0} \frac{r \sin \Delta\theta}{r+\Delta r - r\cos\Delta\theta}.\end{aligned}$$

On replacing $\cos\Delta\theta$ by its equal, $1-2\sin^2\frac{1}{2}\Delta\theta$, and dividing numerator and denominator by $\Delta\theta$, this becomes

$$\tan\psi=\lim_{\Delta\theta \doteq 0} \frac{r\dfrac{\sin\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta}+r\sin\frac{1}{2}\Delta\theta \cdot \dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}}=\frac{r}{\dfrac{dr}{d\theta}}.$$

That is, $$\boldsymbol{\tan\psi = r\frac{d\theta}{dr}}. \tag{1}$$

The angle between the initial line and the tangent at P is usually denoted by ϕ.

It is apparent from Fig. 17 that

$$\phi = \psi + \theta. \tag{2}$$

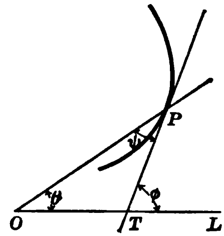

FIG. 17.

NOTE. Results (1) and (2) are true for all polar curves, whatever the figure may be. The student is advised to draw various figures.

61. Lengths of the tangent, normal, subtangent, and subnormal, for any point on a curve: polar coördinates.

In Fig. 18 O is the pole and OL is the initial line. At P any point (r_1, θ_1), on the curve CR, whose equation is $r = f(\theta)$, [or $\phi(r, \theta) = 0$], let the tangent PT and the normal PN be drawn. Produce them to intersect NT, which is drawn through O at right angles to the radius vector OP.

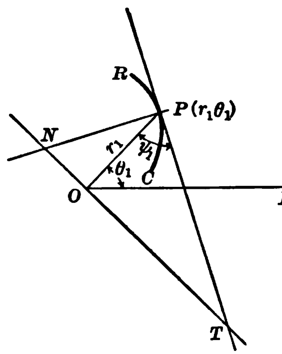

FIG. 18.

The length of the line PT is termed the *length of the tangent* at P; the projection of PT on NT, namely OT, is called the *polar subtangent* for P; the length of PN is termed the *length of the normal* at P; the projection of PN on NT, namely ON, is called the *polar subnormal* for P.

NOTE. In Art. 59 the line used with the tangent and the normal is the x-axis. Here the line so used is *not* the initial line, but the line drawn through the pole at right angles to the radius vector of the point.

In the triangle OPT:

$$OT = OP \tan OPT;$$

i.e. (on removing the subscripts from the letters)

$$\textbf{polar subtangent} = r \tan \psi = r^2 \frac{d\theta}{dr};$$

also, $$TP = OP \sec OPT;$$

i.e. $$\textbf{polar tangent length} = r \sec \psi = r\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}.$$

$$\left[Or:\ TP = \sqrt{\overline{OP}^2 + \overline{OT}^2} = \sqrt{r^2 + r^4\left(\frac{d\theta}{dr}\right)^2} = r\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}.\right]$$

In the triangle OPN:

$$\text{angle } NPO = 90 - \psi;$$

$$ON = OP \tan NPO;$$

i.e. $$\textbf{polar subnormal} = r \cot \psi = \frac{dr}{d\theta};$$

also, $$NP = OP \sec NPO;$$

i.e. $$\textbf{polar normal length} = r \operatorname{cosec} \psi = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

$$\left[Or:\ NP = \sqrt{\overline{OP}^2 + \overline{ON}^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.\right]$$

NOTE. In Fig. 18 r increases as θ increases; accordingly $\frac{d\theta}{dr}$ is positive, and hence the subtangent is positive. Thus when $\frac{d\theta}{dr}$ is positive, the subtangent is measured to the right from an observer at O looking toward P. When r decreases as θ increases, and thus $\frac{d\theta}{dr}$ is negative, the subtangent is measured to the left of the observer looking toward P from O. The student is advised to construct figures for the various cases.

EXAMPLES.

N.B. In the following examples make figures, putting $a = 4$, say. Apply the general results found in these examples to particular concrete cases, *e.g.* $a = 6$ and $\theta = \frac{\pi}{3}$, $a = 2$ and $\theta = \frac{3\pi}{4}$, etc. The angle θ, as used in the equations of the curves, is expressed in radians.

1. In the following curves calculate the lengths of the subnormal, subtangent, normal, and tangent, at any point (r, θ): (1) *The spiral of Archimedes* $r = a\theta$; (2) *the parabolic spiral* or *lituus* $r^2 = a^2\theta$ (*i.e.* $r = a\theta^{\frac{1}{2}}$); (3) *the hyperbolic spiral* (or *the reciprocal spiral*) $r\theta = a$; (4) *the general spiral* $r = a\theta^n$. (The preceding spirals are special cases of this spiral.)

2. From the results in Ex. 1 deduce simple geometrical methods of drawing tangents and normals to the spirals in (1), (2), (3).

3. Do as in Exs. 1, 2, for *the logarithmic spiral* $r = e^{a\theta}$. In this curve each of the lengths specified varies as the radius vector.

4. (*a*) In the spiral of Archimedes $r = a\theta$, show that $\tan\psi = \theta$. Find ψ and ϕ in degrees when angle TOP (Fig. 17) $= 40°$, and when $TOP = 70°$. (*b*) In the curve $r = 4\,\theta$, find ψ and ϕ when $r = 2$.

5. (*a*) In the logarithmic spiral $r = ce^{a\theta}$, show that ψ is constant. This spiral accordingly crosses the radii vectores at a constant angle, and hence is also called *the equiangular spiral.* (*b*) Show that the circle is a special case of the logarithmic spiral, and give the values of ψ and a for this case.

6. In the parabola $r = a\sec^2\dfrac{\theta}{2}$, show that $\phi + \psi = \pi$. Make a practical application of this fact to drawing tangents and normals of this curve.

7. In the cardioid $r = a(1 - \cos\theta)$, show that $\phi = \dfrac{3\,\theta}{2}$, $\psi = \dfrac{\theta}{2}$, subtangent $= 2\,a\tan\dfrac{\theta}{2}\sin^2\dfrac{\theta}{2}$. Apply one of these facts to drawing the tangent and normal at a point on the curve.

62. Applications involving rates. Applications of this kind have already been made in Arts. 26, 37. Rates and differentials have been discussed in Arts. 25–27. It has been seen, Art. 26, Eq. (1), that if $y = f(x)$, then

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}.$$

In words, the rate of change of a function of a variable is equal to the product of the derivative of the function with respect to the variable and the rate of change of the variable. The following principles, which are proved in mechanics, will be useful in some of the examples: (*a*) If a point is moving at a particular moment in such a way that its abscissa x is changing at the rate $\dfrac{dx}{dt}$, and

its ordinate y is changing at the rate $\frac{dy}{dt}$, and if $\frac{ds}{dt}$ denote its rate of motion along its path at that moment, then

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2.$$

(*b*) If a point is moving in a certain direction with a velocity v, the component of this velocity in a direction inclined at an angle α to the first direction, is $v\cos\alpha$.

For instance, if a point is moving so that its abscissa is increasing at the rate 2 feet per second and its ordinate is decreasing at the rate 3 feet per second, it is moving at the rate $\sqrt{2^2+3^2}$, *i.e.* $\sqrt{13}$ feet per second. Again, if a point is moving at the rate of 6 feet per second in a direction inclined 60° to the x-axis, the component of its speed in a direction parallel to the x-axis is $6\cos 60°$, *i.e.* 3 feet per second, and the component parallel to the y-axis is $6\cos 30°$, *i.e.* 5.196 feet per second.

EXAMPLES.

N.B. *Make figures.*

1. If a particle is moving along a parabola $y^2=8x$ at a uniform speed of 4 feet per second, at what rates are its abscissa and its ordinate respectively increasing as it is passing through the point (x, y) and x has successively the values 0, 2, 8, 16 ?

2. A particle is moving along a parabola $y^2=4x$, and, when $x=4$, its ordinate is increasing at the rate of 10 feet per second : find at what rate its abscissa is then changing, and calculate the speed along the curve at that time.

3. A particle is moving along the hyperbola $xy=25$ with a uniform speed 10 feet per second : calculate the rates at which its distances from the axes are changing when it is distant 1 unit and 10 units respectively from the y-axis.

4. A vertical wheel of radius 3 feet is making 25 revolutions per second about an axis through its centre : calculate the vertical and the horizontal components of the velocity, (1) of a point 20° above the level of the axis; (2) of a point 65° above the level of the axis.

5. A point is moving along a cubical parabola $y=x^3$: find (1) at what points the ordinate is increasing 12 times as fast as the abscissa ; (2) at what points the abscissa is increasing 12 times as fast as the ordinate ; (3) how many times as fast as the abscissa is the ordinate growing when $x=10$?

63. Rolle's Theorem.

NOTE 1. ***Progressive and regressive derivative.*** In Art. 22 the derivative of $f(x)$ was defined as

$$\lim_{\Delta x \doteq 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}. \qquad (1)$$

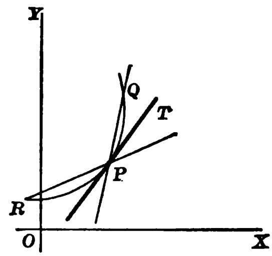

FIG. 19.

The process of evaluating (1) is equivalent to the geometrical process of revolving the chord PQ of the curve $y=f(x)$ about P until Q coincides with P, and thus PQ becomes the tangent PT. If in this curve a chord PR be drawn, and RP be revolved about P until R coincides with P, then RP will finally take the position PT. The slope of the tangent obtained by thus revolving RP is evidently

$$\lim_{\Delta x \doteq 0} \frac{f(x)-f(x-\Delta x)}{\Delta x}; \text{ i.e. } \lim_{\Delta x \doteq 0} \frac{f(x-\Delta x)-f(x)}{-\Delta x}. \qquad (2)$$

It is customary to call (1) *the progressive derivative*, and (2) *the regressive derivative.* In general these derivatives are equal; that is, in general the tangent on the representative curve is the same, whether the secant which is revolved until it assumes a tangential position be drawn forward or backward from the point under consideration. In some cases, however, these derivatives are not equal; such a case is represented at P on Fig. 21 c, where the two revolving secants give two different tangents. In such a case *the derivative is discontinuous at* P, for its value suddenly changes from the slope of TP to the slope of LP.

Theorem. *If a function $f(x)$ and its derivative $f'(x)$ are continuous for all values of x between a and b, and if $f(a)=f(b)$, then $f'(x)=0$ for at least one value of x between a and b.*

Following is a geometrical proof* and representation of this theorem. Let the curve MN (Figs. 20 a, b, c) represent the function $f(x)$.

At M and N let $x=a$ and $x=b$ respectively. Since the ordinates AM and BN are equal, it is evident that there must be at least one point between M and N where the function ceases to increase and begins to decrease, or ceases to decrease and begins

* An analytical discussion will be found in the collateral reading suggested in Note 2, Art. 64.

to increase. There may be several such points, as in Fig. 20 *c*. But at such a point, for instance P, or P_1, or P_2, or P_3, where $x=x_1$ say, $f'(x_1)=0$. (If $f(x)$ is constant, then $f'(x)=0$ at every point.)

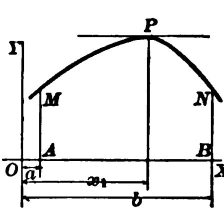

FIG. 20 *a*.

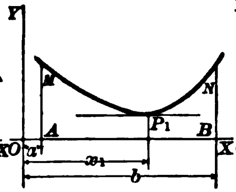

FIG. 20 *b*.

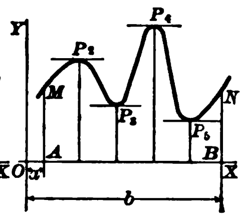

FIG. 20 *c*.

A *special case* of this theorem is that in which $f(a)=0$ and $f(b)=0$. The student may construct the figure for himself by merely moving OX to the position MN. For an application to the theory of equations and for the corresponding algebraic statement of the theorem, see Art. 66 *B*.

NOTE 2. The necessity of the condition relating to continuity is evident from Figs. 21 *a*, *b*, *c*, *d*.

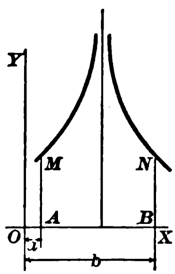

FIG. 21 *a*.

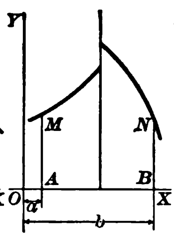

FIG. 21 *b*.

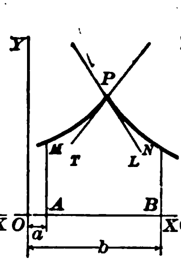

FIG. 21 *c*.

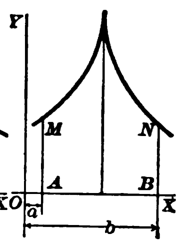

FIG. 21 *d*.

For a value of x between $x=a$ and $x=b$: in Fig. 21 *a*, $f(x)$ is infinite; in Fig. 21 *b*, $f(x)$ is discontinuous; in Fig. 21 *c*, $f'(x)$ is discontinuous; in Fig. 21 *d*, $f'(x)$ is infinite.

64. Theorem of mean value. *If a function $f(x)$ and its derivative $f'(x)$ are continuous for all values of x from $x=a$ to $x=b$, then there is at least one value of x, say x_1, between a and b such that*

$$\frac{f(b)-f(a)}{b-a}=f'(x_1).$$

Following is a geometrical proof* and explanation of this theorem.

Let the curve MN (Fig. 22 a or Fig. 22 b) represent the function $f(x)$. Draw the ordinates AP and BQ at A and B, where

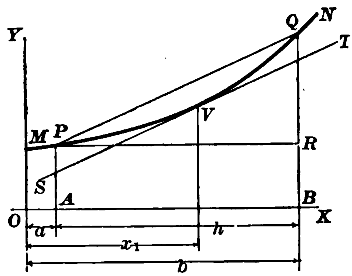

Fig. 22 a.

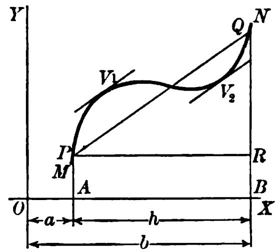

Fig. 22 b.

$x=a$ and $x=b$ respectively. Draw PQ and draw PR parallel to OX. Then

$$AP=f(a),\quad BQ=f(b).$$

Hence
$$RQ=f(b)-f(a),$$

and
$$\tan RPQ=\frac{RQ}{PR}=\frac{f(b)-f(a)}{b-a}.$$

Now the chord PQ and the tangent ST drawn at some point V (or V_1 and V_2) between P and Q evidently must be parallel. At V let $x=x_1$, x_1 thus being between a and b; then $\tan RPQ=f'(x_1)$.

Hence
$$\frac{f(b)-f(a)}{b-a}=f'(x_1). \tag{1}$$

Since x_1 is between a and b, $x_1=a+\theta(b-a)$, in which θ denotes some number between 0 and 1 (*i.e.* $0<\theta<1$). Accordingly, theorem (1) may be expressed

$$f(b)=f(a)+(b-a)f'[a+\theta(b-a)]. \tag{2}$$

If $b-a=h$, then $b=a+h$, and (2) is written

$$f(a+h)=f(a)+hf'(a+\theta h). \tag{3}$$

---

* For an analytical deduction of the theorem of mean value from Rolle's theorem, see Art. 176.

Result (3) has important applications. It is very useful for finding an approximate value of $f(a+h)$ when $f(x)$, a, and h, are given. A closer approximation to the value of $f(a+h)$ can be found by Taylor's formula, Art. 176.

NOTE 1. The necessity for the condition relating to continuity can be made evident by figures similar to Figs. 21 *a*, *b*, *c*, *d*.

NOTE 2. **References for collateral reading** *on Rolle's theorem and the theorem of mean value:* McMahon and Snyder, *Diff. Cal.*, Arts. 59, 66; Lamb, *Calculus*, Arts. 48, 49, 56; Gibson, *Calculus*, §§ 72, 73; Harnack, *Calculus*, Art. 22; Echols, *Calculus*, Chap. V. The last mentioned text has a particularly full and valuable account of these theorems.

EXAMPLES.

1. Find by relation (3) an approximate value of $\sin 32° 20'$ taking $a=32°$: (1) putting $\theta=0$, (2) putting $\theta=1$; and compare the calculated results with that given in the tables.

2. If $f(x)=2x^2-x+5$, find what θ must be in order that relation (3) be satisfied: (1) when $a=3$ and $h=1$; (2) when $a=10$ and $h=2$.

3. Show that for any quadratic function, say $f(x)=lx^2+mx+n$, $f(a+h)$ will be obtained by putting $\theta=\frac{1}{2}$ in relation (3). What geometrical property of the parabola corresponds to this? (*Deduce the value of* θ.)

4. If $f(x)=x^3$, find what θ must be in order that relation (3) be satisfied when $a=3$ and $h=1$. What problem in connection with the cubical parabola $y=x^3$ is the correlative of this?

65. Small errors and corrections: relative error.

If
$$y=f(x), \tag{1}$$
then by Art. 27,
$$dy=f'(x)\cdot dx, \tag{2}$$
in which dx is an assigned change in x. It has been seen (Note 3, Art. 27) that dy is *approximately* the change in y due to dx. An important practical application may be made of this principle. For it follows that if dx be regarded as a small error in the assigned or measured value of x, then dy is an approximate value of the consequent error in y.

The ratio
$$\frac{dy}{y} \text{ or } \frac{f'(x)}{f(x)}\cdot dx \tag{3}$$
is, approximately, *the relative error* or *the proportional error*, *i.e.* the ratio of the error in the value to the value itself.

The approximate values of the correction and relative error may also be deduced from the theorem of mean value. For, if $y = f(x)$, and Δx be an error in x, then $f(x + \Delta x) - f(x)$ is the error in y, *i.e.* the correction that must be applied to y. Now by (3) Art. 64, on putting $a = x$ and $h = \Delta x$,

$$f(x + \Delta x) - f(x) = f'(x + \theta \cdot \Delta x) \cdot \Delta x.$$

Hence, on denoting the error in y by Δy,

$$\Delta y = f'(x) \cdot \Delta x \text{ approximately.}$$

From this the relative error is, approximately, $\dfrac{\Delta y}{y} = \dfrac{f'(x)}{f(x)} \cdot \Delta x.$ (4)

EXAMPLES.

1. The side a of a square is measured, but there is a possible error Δa: find approximately the error in the calculated value of the area. Let A denote the area. Then $A = a^2$; whence $\Delta A = 2\,a \cdot \Delta a$ approximately.

2. If the measured length of the side is 100 inches and this be correct to within a tenth of an inch, find an approximate value of the possible error in the computed area, and an approximate value of the relative error.

In this case, approximately, $\Delta a = 2 \times 100 \times .1 = 20$ square inches. The relative error is, approximately, $\dfrac{20}{100^2}$ or $\dfrac{1}{500}$; that is, 20 square inches in 10,000 square inches, or 1 square inch in 500 square inches.

3. A cylinder has a height h and a radius r inches; there is a possible error Δr inches in r: find by the calculus an approximate value of the possible error in the computed volume. If $h = 10$ inches and the radius is $8 \pm .05$ inches calculate approximately the possible error in the computed volume and the relative error made on taking $r = 8$ inches.

4. Find approximately the error made in the volume of a sphere by making an error Δr in the radius r. The radius of a sphere is said to be 20 inches: give approximate values of the errors made in the computed surface and volume, if there be an error of .1 inch in the length assigned to the radius. Also calculate the relative errors in the radius, the surface, and the volume, and compare these relative errors.

5. Two sides of a triangle are 20 inches and 35 inches. Their included angle is measured and found to be $48° 30'$. It is discovered later that there is an error of $20'$ in this measurement. Find, by the calculus, approximately the error in the computed value of the area of the triangle. Compare the relative errors in the angle and in the area.

6. The exact values of the errors in the computed values in Exs. 1–4 happen to be easily found. Calculate these exact values, and compare with the approximate values already obtained.

7. (1) Two sides, a, b, of a triangle are measured, and also the included angle C: show that the approximate amount of the error in the computed length of the third side c due to a small error ΔC made in measuring C, is

$$\frac{ab \sin C}{\sqrt{a^2 + b^2 - 2\,\overline{ab}}} \cdot \Delta C.$$

(2) Calculate the approximate error in the computed value of the third side in Ex. 5.

66. Applications to algebra.

A. *If $f(x)$ is a rational integral function* of x, and $(x-a)^r$ is a factor of $f(x)$, then $(x-a)^{r-1}$ is a factor of $f'(x)$.*

Let $$f(x) = (x-a)^r \phi(x).$$

Then $$f'(x) = r(x-a)^{r-1}\phi(x) + (x-a)^r \phi'(x)$$
$$= (x-a)^{r-1}[r\phi(x) + (x-a)\phi'(x)].$$

It follows from this theorem that *if $f(x)$ is a rational integral function of x, and a is an r-tuple (or r-fold) root of the equation $f(x)=0$, then a is an $(r-1)$-tuple root of the equation $f'(x)=0$.* This theorem may be employed in finding the multiple roots of an equation.

Ex. 1. Solve $x^3 - 2x^2 - 15x + 36 = 0$ by trying for equal roots.

The derived equation is $3x^2 - 4x - 15 = 0$,

i.e. $$(3x+5)(x-3) = 0.$$

Trial will show that $(x-3)^2$ is a factor of $x^3 - 2x^2 - 15x + 36$, and the first equation is $(x-3)^2(x+4) = 0$. The roots are thus: 3, 3, -4.

NOTE. *The multiple roots of $f(x) = 0$ will be revealed on finding the highest common factor of $f(x)$ and $f'(x)$.*

Ex. 2. Solve the following equations:

(1) $$3x^3 + 4x^2 - x - 2 = 0;$$

(2) $$4x^3 + 16x^2 + 21x + 9 = 0;$$

(3) $$x^4 - 11x^3 + 44x^2 - 76x + 48 = 0;$$

(4) $$8x^4 + 4x^3 - 62x^2 - 61x - 15 = 0;$$

(5) $$x^5 + x^4 - 13x^3 - x^2 + 48x - 36 = 0.$$

Ex. 3. Find the condition that $x^n - px^2 + r = 0$ may have equal roots.

---

* A rational integral function of x is a function in which x has only positive integral exponents and does not appear in the denominator of a fraction; *e.g.* $x^2 - \frac{1}{2}x + 2$, $ax^n + bx^{n-1} + \cdots + mx + p$, if n is a positive integer.

B. *An important application of Rolle's Theorem may be made to the theory of equations.* According to the theorem, geometrically, the slope of a curve $y = f(x)$ is zero once at least, between the

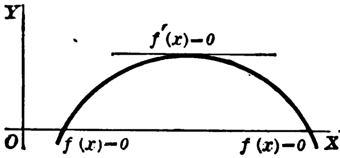

FIG. 23 *a*.

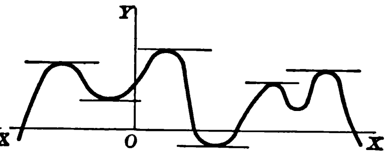

FIG. 23 *b*.

points where the curve crosses the x-axis. Hence, **at least one real root of the equation $f'(x) = 0$ lies between any two real roots of the equation $f(x) = 0$.** (In the theory of equations this is called Rolle's Theorem.*)

NOTE. According to this principle r real roots of an equation $f(x) = 0$ have at least $(r - 1)$ roots of $f'(x) = 0$ between them. Now, if the r roots coalesce and thus make an r-tuple root, the $(r - 1)$ roots must also coalesce and thus make an $(r - 1)$-tuple root of $f'(x) = 0$. (Compare A above.)

Ex. Verify Rolle's Theorem in each of the following equations $f(x) = 0$; also sketch the curve $y = f(x)$:

(1) $x^2 + x - 6 = 0$; (2) $x^3 + 2x^2 - 5x - 6 = 0$.

67. Geometric derivatives and differentials.

(*a*) **Derivative and differential of an area: rectangular coördinates.** Let PQ be an arc of the curve $y = f(x)$. Take any point on PQ, $V(x, y)$ say, and take $T(x + \Delta x, y + \Delta y)$. Construct the rectangles VN and TM as shown in Fig. 24. Draw the ordinate BP, and let the area of $BPVM$ be denoted by A; then the area of $MVTN$ may be denoted by ΔA.

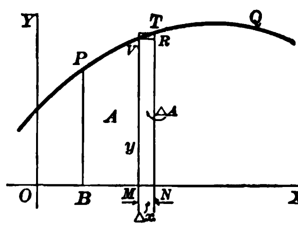

FIG. 24.

Now, rectangle $VN <$ $MVTN <$ rectangle MT;

i.e. $y \cdot \Delta x < \Delta A < (y + \Delta y)\,\Delta x$.

Hence, on division by Δx, $y < \dfrac{\Delta A}{\Delta x} < y + \Delta y$. (1)

* After *Michel Rolle* (1652–1719).

On letting Δx approach zero, these quantities (Arts. 18, 22, 23) approach the values y, $\frac{A}{dx}$, y, respectively.

$$\therefore \frac{dA}{dx} = y. \tag{2}$$

That is, *the derivative of the area BPVM* with respect to the abscissa x of V, is the measure of the ordinate of V. On denoting this measure by y, result (2) means (Art. 26) that the area $BPVM$ is increasing y times as fast as the abscissa of V. From (2) it follows by Art. 27 that

$$dA = y \cdot dx. \tag{3}$$

That is, *the differential of the area BPVM* is the area of a rectangle whose height is the ordinate MV and whose base is dx, the differential of the abscissa of V.

Ex. 1. Find the derivative of the area between the x-axis and the curve $y = x^3$, with respect to the abscissa: (*a*) at the point whose abscissa is 2; (*b*) at the point whose abscissa is 4.

$$(a)\quad \frac{dA}{dx} = y, \text{(where } x = 2,) = 2^3 = 8. \tag{4}$$

$$(b)\quad \frac{dA}{dx} = y, \text{(where } x = 4,) = 4^3 = 64. \tag{5}$$

These results mean that, if an ordinate, like VM in the figure, is moving to the right or left at a certain rate, the area of the figure bounded on one side by that ordinate is changing, in case (*a*) at 8 times that rate, and in case (*b*) at 64 times that rate.

Ex. 2. Find the differentials in Ex. 1 (*a*) and (*b*), when $dx = .1$ inch. Show these differentials on a drawing.

By (3), (4), and (5), in case (*a*), $dA = .8$ square inch; in case (*b*) $dA = 6.4$ square inches.

NOTE. The area .8 square inch is nearly the actual increase in area between the curve and the x-axis when the ordinate moves from $x = 2$ to $x = 2.1$; and 6.4 square inches is nearly the increase in this area when the ordinate moves from $x = 4$ to $x = 4.1$. These increases are calculated in Ex. 16, Art. 111.

It is evident that the smaller dx is taken, the more nearly will the differential of the area become equal to the actual increase of the area between the curve and the x-axis.

Ex. 3. Show that the y-derivative of an area between the curve and the y-axis is x. Thence deduce that the y-differential of this area is $x\,dy$, and make a figure showing this differential area.

Ex. 4. In the case of the cubical parabola $y = x^3$ find $\frac{dA}{dx}$ and $\frac{dA}{dy}$; then calculate the differential of the area between this curve and the x-axis at the point (2, 8), taking $dx = .2$. Also calculate the differential of the area between this curve and the y-axis at the same point, taking $dy = .2$. Show these differentials in a figure.

(b) Derivative and differential of an area: polar coördinates. Let PQ be an arc of the curve $f(r, \theta) = 0$. On PQ take any point $V(r, \theta)$, and take the point $W(r + \Delta r, \theta + \Delta\theta)$. About O describe a circular arc VN intersecting OW in N, and describe a circular arc WM intersecting OV in M. Then $NW = \Delta r$, and $VOW = \Delta\theta$. Also (Pl. Trig., p. 175), area sector $VON = \frac{1}{2} r^2\Delta\theta$, and area sector $MOW = \frac{1}{2}(r + \Delta r)^2\Delta\theta$.

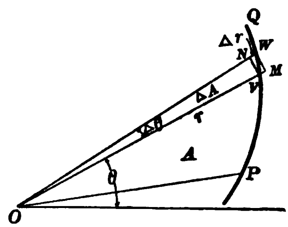

FIG. 25.

Draw OP. Let the area of POV be denoted by A; then the area of VOW may be denoted by ΔA.

Now, area $VON <$ area $VOW <$ area MOW;

i.e.
$$\tfrac{1}{2} r^2\Delta\theta < \Delta A < \tfrac{1}{2}(r + \Delta r)^2\Delta\theta.$$

$$\therefore \tfrac{1}{2} r^2 < \frac{\Delta A}{\Delta\theta} < \tfrac{1}{2}(r + \Delta r)^2.$$

On letting $\Delta\theta$ approach zero, these quantities (Arts. 18, 22, 23) approach the values

$$\tfrac{1}{2} r^2, \ \frac{dA}{d\theta}, \ \tfrac{1}{2} r^2, \text{ respectively.}$$

$$\therefore \frac{dA}{d\theta} = \frac{1}{2} r^2. \tag{1}$$

Result (1) means that, if the radius vector is revolving at a certain rate, the area passed over by the radius vector, when its length is r, is increasing at a rate which is $\frac{1}{2} r^2$ (*i.e.* the *number*) times the rate of revolution.

It follows from (1) and Art. 27 that

$$dA = \frac{1}{2} r^2 d\theta. \tag{2}$$

Ex. 5. Show that in the case of the circle the differential of the area swept over by a revolving radius is the additional area passed over.

Ex. 6. In the spiral of Archimedes $r = 2\theta$ find the derivative of the area swept over by the radius vector, with respect to θ. Calculate the differential of this area when: (1) $\theta = 30°$ and $d\theta = 30'$; (2) $r = 2$ and $d\theta = 1°$. Make a figure showing these differentials.

Ex. 7. In the cardioid $r = 4(1 - \cos\theta)$ find the θ-derivative of the area. Calculate the differential of the area when: (1) $\theta = 60°$ and $d\theta = 1°$; (2) $\theta = 0$ and $d\theta = 2°$; (3) $\theta = 330°$ and $d\theta = 1°$. Make a figure showing these differentials.

(c) Derivative and differential of the length of a curve: rectangular coördinates. Let PQ be an arc of the curve $y = f(x)$. On PQ take any point $M(x, y)$, and take the point $N(x + \Delta x, y + \Delta y)$; and draw the chord MN. On denoting the length of the arc PM by s, the length of the arc MN may be denoted by Δs.

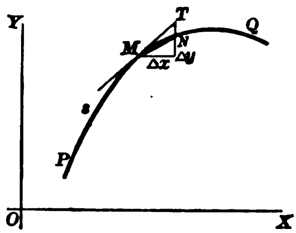

FIG. 26.

Now it follows from Art. 19, Ex. 6, Note, and the theory of limits (Arts. 20, 21), that

$$\lim_{\Delta x \doteq 0} \frac{\text{arc } MN}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\text{chord } MN}{\Delta x}; \tag{1}$$

i.e.
$$\lim_{\Delta x \doteq 0} \frac{\Delta s}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x} = \lim_{\Delta x \doteq 0} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

That is,
$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2}$$

Similarly,
$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{3}$$

From (2), (3), and Art. 27,
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx; \tag{4}$$

and
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy. \tag{5}$$

Ex. 8. Show that for a given dx and the actual derivative $\frac{dy}{dx}$ at M, the second member of (4) gives the length of the intercept of the tangent, namely, MT. Show that for a given dx, and using dy to denote the exact corresponding change in the ordinate, the second members in (4) and (5) give the length of the chord of the arc, namely, the line MN.

NOTE. It is shown in Art. 137 how to find the length of the arc MN corresponding to an increment dx in x. The smaller dx is, the more nearly will MT, arc MN, and chord MN, become equal to one another. See Ex. 6, Art. 19.

Ex. 9. (1) Calculate the x-derivative and the y-derivative of the arc of the parabola $y^2 = 4ax$. (2) Find the x-derivative of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 10. In the cubical parabola $y = x^3$ calculate the differential of the arc at the point (2, 8) when: (1) $dx = .2$; (2) $dy = .1$. Show these differentials in a figure. (The actual increments of the arcs can be computed by Art. 137.)

(d) Derivative and differential of the length of a curve: polar coördinates. Let PQ be an arc of the curve $f(r, \theta) = 0$. On PQ take any point $V(r, \theta)$, and take $W(r + \Delta r, \theta + \Delta\theta)$. Denote the length of PV by s; then the length of VW may be denoted by Δs. Draw the chord VW.

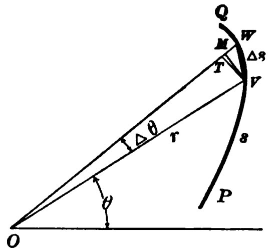

FIG. 27.

Now, as in (c),

$$\lim_{\Delta\theta \doteq 0} \frac{\text{arc } VW}{\Delta\theta} \left(i.e.\ \frac{ds}{d\theta}\right) = \lim_{\Delta\theta \doteq 0} \frac{\text{chord } VW}{\Delta\theta}. \quad (1)$$

About O describe a circular arc VM intersecting OW in M, and draw VT at right angles to OW. Then angle $VOW = \Delta\theta$, and $MW = \Delta r$.

$$\therefore\ TW = OW - OT = r + \Delta r - r\cos\Delta\theta, \text{ and } VT = r\sin\Delta\theta.$$

$$\therefore \text{ chord } VW = \sqrt{(VT)^2 + (TW)^2} = \sqrt{(r\sin\Delta\theta)^2 + [r(1 - \cos\Delta\theta) + \Delta r]^2}.$$

$$\therefore\ \frac{\text{chord } VW}{\Delta\theta} = \sqrt{\left(r\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left[r \cdot \frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta} \cdot \sin\tfrac{1}{2}\Delta\theta + \frac{\Delta r}{\Delta\theta}\right]^2}. \quad (2)$$

$$\therefore\ \lim_{\Delta\theta \doteq 0} \frac{\text{chord } VW}{\Delta\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2},$$

since, if $\Delta\theta \doteq 0$, $\dfrac{\sin\Delta\theta}{\Delta\theta} \doteq 1$, $\dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta} \doteq 1$, and $\sin\frac{1}{2}\Delta\theta \doteq 0$.

Hence, by (1),
$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \quad (3)$$

On multiplying each member of (2) by $\dfrac{\Delta\theta}{\Delta r}$, and then letting $\Delta\theta$, and consequently Δr, approach zero, it will be found that

$$\frac{ds}{dr} = \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}. \quad (4)$$

From (3), (4), and definition Art. 27,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta, \quad (5)$$

and
$$ds = \sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1} \cdot dr. \quad (6)$$

Ex. 11. Find the derivative of the arc of the spiral of Archimedes $r = a\theta$: (1) with respect to the angle; (2) with respect to the radius vector.

Ex. 12. Calculate the differential of the arc of the Archimedean spiral $r = 2\theta$ when $\theta = 2$ radians and $d\theta = 1°$. Make a figure. (The actual increment of the arc can be computed by Art. 138.)

(e) Derivative and differential of the volume of a surface of revolution. Let PQ be an arc of the curve $y = f(x)$. On PQ take any point $L(x, y)$, and take the point $M(x + \Delta x, y + \Delta y)$. On letting V denote the volume obtained by revolving arc PL about OX, the volume obtained by revolving arc LM may be denoted by ΔV. Through L and M draw the lines shown in the figure.

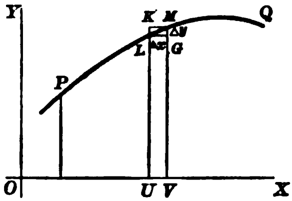

FIG. 28.

The volume obtained by revolving arc LM about the x-axis is greater than the volume obtained by revolving LG, and is less than the volume obtained by revolving KM. That is,

$$\pi \cdot \overline{UL}^2 \cdot LG < \Delta V < \pi \cdot \overline{VM}^2 \cdot KM;$$

i.e.
$$\pi y^2 \cdot \Delta x < \Delta V < \pi \cdot (y + \Delta y)^2 \cdot \Delta x.$$

$$\therefore \pi y^2 < \frac{\Delta V}{\Delta x} < \pi (y + \Delta y)^2. \quad (1)$$

On letting Δx approach zero, the three numbers in (1) become

$$\pi y^2, \frac{dV}{dx}, \pi y^2, \text{ respectively.}$$

Hence,
$$\frac{dV}{dx} = \pi y^2. \quad (2)$$

From (2) and Art. 27
$$dV = \pi y^2 \cdot dx. \quad (3)$$

If PQ had been revolved about the y-axis, then

$$\frac{dV}{dy} = \pi x^2, \text{ and } dV = \pi x^2 \cdot dy. \quad (4)$$

NOTE. According to (3), for a given differential dx the corresponding differential of the volume is the volume of a cylinder of radius y and height dx. The smaller dx is, the more nearly does this volume become equal to the actual increment, due to dx, in the volume of the solid of revolution.

Ex. 13. Derive the results in (4).

Ex. 14. (1) Find the x-derivative of the volume generated by the revolution of the parabola $y = x^2$ about the x-axis. (2) Find the y-derivative of the volume generated by the revolution of this curve about the y-axis.

Ex. 15. (1) Calculate the differential of the volume in Ex. 14 (1), taking $dx = .1$ at the point where $x = 2$. (2) Thus also in Ex. 14 (2), taking $dy = .2$ at the point where $x = 4$. (The actual increment in the volume of the solid due to changes dx and dy can be computed by Art. 112.)

(f) Derivative and differential of the area of a surface of revolution. Let PQ be an arc of the curve $y = f(x)$. On PQ take any point, say $L(x, y)$, and take the point $M(x + \Delta x, y + \Delta y)$. Let S denote the area of the surface generated by revolving arc PL about OX; then the area generated by revolving arc LM about OX may be denoted by ΔS. There is evidently a straight line whose length is equal to the length of the arc LM. Through L and M draw the lines LM' and ML' parallel to OX and equal in length to the arc LM. (LM may be supposed to be a piece of wire, LM' the same piece of wire when it is stretched out in a horizontal straight line from L, and ML' the same piece of wire when it is stretched out in a horizontal line from M.) The surface obtained by revolving the arc LM about OX is greater than the surface obtained by revolving LM'; for, with the exception of the point L, each point on LM has a greater ordinate than the corresponding point in the line LM', and consequently a greater radius of swing. Similarly, the surface obtained by revolving LM is less than the surface obtained by revolving ML'. That is,

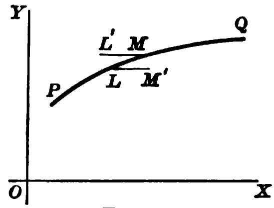

FIG. 29.

$$2\pi y \cdot LM' < \text{surface generated by } LM < 2\pi(y + \Delta y) \cdot L'M;$$

i.e.
$$2\pi y \cdot \text{arc } LM < \Delta S < 2\pi(y + \Delta y) \cdot \text{arc } LM. \tag{1}$$

$$\therefore\ 2\pi y \frac{\text{arc } LM}{\Delta x} < \frac{\Delta S}{\Delta x} < 2\pi(y + \Delta y)\frac{\text{arc } LM}{\Delta x}. \tag{2}$$

On letting Δx approach zero, the three numbers in (2), by Arts. 20, 22, 23, 67c, take the values

$$2\pi y \frac{ds}{dx},\ \frac{dS}{dx},\ 2\pi y \frac{ds}{dx}, \text{ respectively;}$$

and hence
$$\frac{dS}{dx} = 2\pi y \frac{ds}{dx}. \tag{3}$$

On dividing the members in (1) by Δy, and letting Δy approach zero,

$$\frac{dS}{dy} = 2\pi y \frac{ds}{dy}. \tag{4}$$

Similarly, if arc PQ revolve about the y-axis,

$$\frac{dS}{dx} = 2\pi x \frac{ds}{dx} \quad (5), \quad \text{and} \quad \frac{dS}{dy} = 2\pi x \frac{ds}{dy}. \tag{6}$$

From (3), (4), and Art. 67 (c) [(2), (3)],

$$\frac{dS}{dx} = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2},\quad \frac{dS}{dy} = 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{7}$$

Similarly, in case of revolution about the y-axis, from (5) and (6),

$$\frac{dS}{dx}=2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2},\ \frac{dS}{dy}=2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}. \tag{8}$$

Results (3), (4), (7), show that, **for a curve revolving about the x-axis,**

$$dS=2\pi y\cdot ds=2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy; \tag{9}$$

and (5), (6), (8), show that, **for a curve revolving about the y-axis,**

$$dS=2\pi x\cdot ds=2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy. \tag{10}$$

Ex. 16. Derive results (5), (6), (8), and (10).

Ex. 17. Find the x-derivative and the y-derivative of each of the surfaces described in Ex. 14.

Ex. 18. Calculate the differentials of the surfaces described in Ex. 15. Make figures showing these differentials. (The actual increments of the surfaces can be computed by Art. 139.)

Ex. 19. Find $\frac{ds}{dx}$, $\frac{dA}{dx}$, $\frac{dV}{dx}$, $\frac{dS}{dx}$, for the ellipse $b^2x^2+a^2y^2=a^2b^2$. For a given differential of x, draw figures showing the corresponding differentials of s, A, V, and x.

Ex. 20. Find $\frac{ds}{d\theta}$ for $r^2=a^2\cos 2\theta$, $r=a\cos\theta$, $r=ae^{\theta\cot\alpha}$, $r=a(1+\cos\theta)$.

Ex. 21. If ϕ denote the eccentric angle of the ellipse in Ex. 19, show that

$$\frac{ds}{d\phi}=a\sqrt{1-e^2\cos^2\phi},\ e\text{ being the eccentricity.}$$

CHAPTER VI.

SUCCESSIVE DIFFERENTIATION.

N.B. Article 68 contains all that the beginner will find necessary concerning successive differentiation for the larger part of the remaining chapters. Accordingly, the reading of Arts. 69–72 may be deferred until later.

68. Successive derivatives. As observed in many of the preceding examples, the derivative of a function of x is, in general, also a function of x. This derivative, which may be called the *first* derived function, or *the first derivative* (of the function), may itself be differentiated; the result is accordingly called the *second* derived function, or *the second derivative* (of the original function). If the second derivative is differentiated, the result is called the *third* derived function, or *the third derivative;* and so on. If the operation of differentiation is performed on a function n times in succession, the final result is called the nth derived function, or *the* nth *derivative,* of the function.

Ex. If the function is x^4, then its first derivative is $4x^3$; its second derivative is $12x^2$; its third derivative is $24x$; its fourth derivative is 24; its fifth and its succeeding derivatives are all zero.

Notation. (*a*) If y denote the function of x, then

the first derivative, namely $\dfrac{d}{dx}(y)$, is denoted by $\dfrac{dy}{dx}$ (Art. 23);

the second derivative, namely $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, is denoted by $\dfrac{d^2y}{dx^2}$;

the third derivative, namely $\dfrac{d}{dx}\left[\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)\right]$, is denoted by $\dfrac{d^3y}{dx^3}$;

and so on. On this plan of writing,

the nth derivative is denoted by $\dfrac{d^ny}{dx^n}$.

In this notation the integers $2, 3, \cdots, n$, are not exponents; these integers merely indicate the number of times that the function y is to be differentiated successively with respect to x.

(*b*) The letter D is frequently used to denote both the operation and the result of the operation indicated by the symbol $\frac{d}{dx}$. (See Art. 23.) The successive derivatives of y are then Dy, $D(Dy)$, $D[D(Dy)]$, $\cdots$; these are respectively denoted by

$$Dy,\ D^2y,\ D^3y,\ \cdots,\ D^ny.$$

Sometimes there is an indication of the variable with respect to which differentiation is performed; thus

$$D_xy,\ D_x^2y,\ D_x^3y,\ \cdots,\ D_x^ny.$$

NOTE. Here n is not an exponent; D^ny does not mean $(Dy)^n$. (*E.g.* see Ex., p. 107.) D^ny is called the derivative of the nth **order.**

(*c*) Instead of the symbols shown in (*a*) and (*b*), for the successive derivatives of y, the following are sometimes used, namely,

$$y',\ y'',\ y''',\ \cdots,\ y^{(n)}.$$

(*d*) If the function be denoted by $\phi(x)$, its first, second, third, $\cdots$, and nth derivatives (with respect to x) are generally denoted by

$$\phi'(x),\ \phi''(x),\ \phi'''(x),\ \cdots,\ \phi^{(n)}(x) \text{ or } \phi^n(x), \text{ respectively.}$$

NOTE 1. In this book notation (*a*) is most frequently used. The symbol D is very convenient, and is especially useful in certain investigations. See Byerly's *Diff. Cal.*, Lamb's *Calculus*, Gibson's *Calculus* (in particular § 67). For an exposition of simple elementary properties of the symbol D also see Murray's *Differential Equations* (edition 1898), Note K, page 208.

NOTE 2. Instead of the accent notation in (*c*), the 'dot'-age notation,

$$\dot{y},\ \ddot{y},\ \dddot{y},\ \cdots$$

is sometimes used, particularly in physics and mechanics.

NOTE 3. **Geometrical meaning of $\frac{d^2y}{dx^2}$.** It has been seen in Arts. 25, 26, that $\frac{dy}{dx}$, *i.e.* $\frac{d}{dx}(y)$, denotes the rate of change of y, the ordinate of the curve, compared with the rate of change of the abscissa x; this may be simply denoted as the x-rate of change of the ordinate. Similarly $\frac{d^2y}{dx^2}$, *i.e.* $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, is *the rate of change of the slope* $\frac{dy}{dx}$ *of a curve* compared with the rate of change of the abscissa x, or, simply, the x-rate of change of the slope.

On a straight line, for instance, the slope is constant, and hence the x rate of change of the slope is zero. This is also apparent analytically. For, if $y = mx + c$ is the equation of the line, then $\frac{dy}{dx} = m$, and hence $\frac{d^2y}{dx^2} = 0$.

NOTE 4. **Physical meaning of $\frac{d^2s}{dt^2}$.** In Art. 25 it has been seen that if s denotes a varying distance along a straight line, $\frac{ds}{dt}$, *i.e.* $\frac{d}{dt}(s)$, denotes the rate of change of this distance, *i.e.* a velocity. Similarly $\frac{d^2s}{dt^2}$, *i.e.* $\frac{d}{dt}\left(\frac{ds}{dt}\right)$, denotes the rate of change of this velocity. Rate of change of velocity is called *acceleration*. For instance, if a train is going at the rate of 30 miles an hour, and half an hour later is going at the rate of 40 miles an hour, its velocity has increased by '10 miles an hour' in half an hour, *i.e.* as usually expressed, its acceleration is 10 miles per hour per half an hour. Again, it is known that if s is the distance through which a body falls from rest in t seconds, $s = \frac{1}{2}gt^2$. Hence $\frac{ds}{dt} = gt$; accordingly, $\frac{d^2s}{dt^2} = g$. That is, the acceleration of a falling body is 'g feet per second' per second. (See text-books on Kinematics, Dynamics, and Mechanics, for a discussion on acceleration.)

EXAMPLES.

1. Find the second x-derivative of: (i) $x \tan^{-1} x$; (ii) $4x^2 - 9x + \frac{3}{x} - \sqrt{x}$; (iii) $\tan x + \sec x$; (iv) x^x.

2. Find $D_x^3 y$, when: (i) $y = (x^2 + a^2) \tan^{-1}\frac{x}{a}$; (ii) $y = \log(\sin x)$.

3. Find $\frac{d^4y}{dx^4}$, when: (i) $y = \sin^{-1} x$; (ii) $y = \frac{1}{1 + x^2}$.

4. Find $D_x^6 y$, when: (i) $y = x^4 \log x$; (ii) $y = e^x \cos x$.

5. Find $\frac{d^2y}{dx^2}$, when $xy^2 + 3x + 5y = 0$.

By Art. 56, $$\frac{dy}{dx} = -\frac{y^2 + 3}{2xy + 5}. \qquad (1)$$

On differentiation, $$\frac{d^2y}{dx^2} = -\frac{(2xy + 5)\,2y\frac{dy}{dx} - (y^2 + 3)\left(2y + 2x\frac{dy}{dx}\right)}{(2xy + 5)^2}.$$

On substituting the value of $\frac{dy}{dx}$, and reducing,

$$\frac{d^2y}{dx^2} = \frac{2(y^2 + 3)(3xy^2 + 10y - 3x)}{(2xy + 5)^3}. \qquad (2)$$

6. (i) In the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ calculate $D_x^2 y$. (ii) Given $y^2 + y = x^2$, find $D_x^3 y$.

7. Show that the point $(\frac{1}{2}, \frac{1}{2})$ is on the curve $\log(x+y) = x - y$. Show that at this point $\frac{dy}{dx} = 0$, and $\frac{d^2y}{dx^2} = \frac{1}{2}$.

8. What are the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$: (i) at the point (2, 1) on the ellipse $7x^2 + 10y^2 = 38$; (ii) at the point (3, 5) on the parabola $y^2 = 4x + 13$.

9. Calculate $\frac{d^2y}{dx^2}$ for the cycloid in Art. 43, Ex. 6. Compute it when $a = 8$ and $\theta = \frac{\pi}{3}$.

10. Verify the following: (i) if $y = a\sin x + b\cos x$, $\frac{d^2y}{dx^2} + y = 0$; (ii) if $u = (\sin^{-1}x)^2$, $(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} = 2$; (iii) if $y = a\cos(\log x) + b\sin(\log x)$, $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

11. Show that if $u = y^2 \log y$, and $y = f(x)$, $\frac{d^2u}{dx^2} = (2\log y + 3)\left(\frac{dy}{dx}\right)^2 + y(2\log y + 1)\frac{d^2y}{dx^2}$.

12. Find $\frac{d^2y}{dx^2}$ in the following cases: $y = 4x^3 + 2x - 3$, $y = 4x^3 + 4x + 2$, $y = 4x^3 + 5x - 4$, $y = 4x^3 + cx + k$.

13. Given that $\frac{d^2y}{dx^2} = 3x + 2$, find the most general expression for $\frac{dy}{dx}$; then find the most general expression for y.

14. A curve passes through the point (2, 3) and its slope there is 1; at any point on this curve $\frac{d^2y}{dx^2} = 2x$: find its equation and sketch the curve.

15. At any point on a certain curve $\frac{d^2y}{dx^2} = 8$; the curve passes through the origin and touches the line $y = x$ there; find its equation and sketch the curve.

16. (1) In the case of simple harmonic motion, Ex. 13 (p. 83), show that the speed of Q is changing at a rate which varies as the distance of Q from the centre of the circle. (2) What is the acceleration of the velocity of the boat in Ex. 18, Art. 37 ?

17. In Ex. 14 (p. 83), calculate the rate at which Q is changing its speed when Q is: (i) at an extremity of the diameter; (ii) 12 inches from the centre; (iii) 6 inches from the centre; (iv) at the centre.

18. A body moving vertically has an acceleration or a retardation of g feet per second due to gravitation, g being a number whose approximate value is 32.2 : find the most general expression for the distance of the moving point from a fixed point in its line of motion, after t seconds. Explain the physical meaning of the constants that are introduced in the course of integration.

19. A body is projected vertically upwards with an initial velocity of 500 feet per second: find how long it will continue to rise, and what height it will reach, if the resistance of the air be not taken into account.

20. A rifle ball is fired through a three-inch plank, the resistance of which causes an unknown constant retardation of its velocity. Its velocity on entering the plank is 1000 feet a second, and on leaving the plank is 500 feet a second. How long does it take the ball to traverse the plank? (Byerly, *Problems in Differential Calculus.*)

69. The nth derivative of some particular functions. In a few cases the nth derivative of a function can be found. This is done by differentiating the function a few times in succession, and thereby being led to see a connection between the successive derivatives.

EXAMPLES.

1. Let $$y = x^r.$$

Then
$$Dy = rx^{r-1};$$
$$D^2y = r(r-1)x^{r-2};$$
$$D^3y = r(r-1)(r-2)x^{r-3}.$$

From this it is evident that

$$D^ny = r(r-1)(r-2)\cdots(r-n+1)x^{r-n}.$$

Show that $D^nx^n = n\,!$

2. Find the nth derivative of the following functions:

$$(a)\ e^x; \qquad (b)\ a^x; \qquad (c)\ e^{ax}; \qquad (d)\ a^{bx}.$$

3. Show that the nth derivative of $\sin x$ is $\sin\left(x + \frac{n\pi}{2}\right)$.

$\left[\text{Suggestion: } \cos z = \sin\left(z + \frac{\pi}{2}\right).\right]$

4. Find the nth derivatives of (a) $\cos x$; (b) $\sin ax$; (c) $\cos ax$.

5. Find the nth derivatives of $\log x$, $\log(x-2)^2$.

6. Find the nth derivatives of $\frac{1}{x}$, $\frac{1}{1+x}$, $\frac{2}{3-x}$, $\frac{a}{(b+cx)^m}$.

7. Find the nth derivatives of $\frac{2}{1-x^2}$, $\frac{2x}{1-x^2}$.

[Suggestion: Take the partial fractions.]

70. Successive differentials. In Art. 27 it has been shown that if

$$y = f(x), \tag{1}$$

then $$dy = f'(x)dx. \tag{2}$$

The differential in (2) is, in general, also a function of x; and its differential may be required. In obtaining successive differentials it is usual to give a *constant* differential increment dx to x. Then (Art. 27), on taking the differentials of the members in (2),

$$d(dy) = d[f'(x)dx] = [f''(x)dx]\,dx. \tag{3}$$

On taking the differentials of the members of (3),

$$d\{d(dy)\} = d\{[f''(x)dx]\,dx\} = f'''(x)dx \cdot dx \cdot dx. \tag{4}$$

It is customary to denote results (3) and (4) thus:

$$d^2y = f''(x)dx^2 \text{ and } d^3y = f'''(x)dx^3.$$

In this notation **the nth differential is written**

$$\boldsymbol{d^ny = f^n(x)dx^n},$$

in which $f^n(x)$ denotes the nth derivative of $f(x)$, and dx^n denotes $(dx)^n$.

71. The successive derivatives of y with respect to x when both are functions of a third variable, t say.

By Art. 26, Note 1, $$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} \quad \text{[by the principle in Art. 34, Eq. (2)]}$$

$$= \frac{\dfrac{dx}{dt} \cdot \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} \cdot \dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^2} \cdot \frac{dt}{dx} = \frac{\dfrac{dx}{dt} \cdot \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} \cdot \dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3}.$$

The method of obtaining the higher derivatives is similar. Thus,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) \cdot \frac{dt}{dx} = \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right) \div \frac{dx}{dt}.$$

And, in general,

$$\frac{d^ny}{dx^n}=\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)\cdot\frac{dt}{dx}=\frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)\div\frac{dx}{dt}.$$

Ex. 1. See Ex. 9, Art. 68.

Ex. 2. Find D_xy and D_x^2y when $x=a-b\cos\theta$ and $y=a\theta+b\sin\theta$.

72. Leibnitz's theorem. This theorem gives a formula for the nth derivative of the product of two variables. Suppose that u and v are functions of x, and put $y=uv$.

Then, on performing successive differentiations,

$$Dy\;=u\cdot Dv+v\cdot Du;$$

$$D^2y=u\cdot D^2v+2\,Du\cdot Dv+v\cdot D^2u;$$

$$D^3y=u\cdot D^3v+3\,Du\cdot D^2v+3\,D^2u\cdot Dv+v\cdot D^3u;$$

$$D^4y=u\cdot D^4v+4\,Du\cdot D^3v+6\,D^2u\cdot D^2v+4\,D^3u\cdot Dv+v\cdot D^4u.$$

Thus far the numerical coefficients in these derivatives are the same as the numerical coefficients in the expansions $(u+v)$, $(u+v)^2$, $(u+v)^3$, and $(u+v)^4$ respectively, and the orders of the derivatives of u and v are the same as the exponents of u and v in those binomial expansions. Now suppose that these laws (for the numerical coefficients and the orders) hold in the case of the nth derivative of uv; that is, suppose that

$$\begin{aligned}D^n(uv)=u\cdot D^nv+nDu\cdot D^{n-1}v+\frac{n(n-1)}{1\cdot 2}D^2u\cdot D^{n-2}v+\cdots\\ +\frac{n(n-1)\cdots(n-r+2)}{1\cdot 2\cdots(r-1)}D^{r-1}u\cdot D^{n-r+1}v\\ +\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}D^ru\cdot D^{n-r}v+\cdots+v\cdot D^nu.\quad(1)\end{aligned}$$

Then these laws for the coefficients and the orders hold in the case of the $(n+1)$th derivative of uv. For differentiation of both members of (1) gives

$$\begin{aligned}D^{n+1}(uv)=u\cdot D^{n+1}v+(n+1)Du\cdot D^nv+\frac{(n+1)n}{1\cdot 2}D^2u\cdot D^{n-1}v+\cdots\\ +\frac{(n+1)n(n-1)\cdots(n-r+2)}{1\cdot 2\cdots(r-1)r}D^ru\cdot D^{n-r+1}v+\cdots+v\cdot D^{n+1}u.\end{aligned}$$

Hence, if formula (1) is true for the nth derivative of uv, a similar formula holds for the $(n+1)$th derivative. But, as shown above, formula (1) is true for the first, second, third, and fourth derivatives of uv; hence it is true for the fifth, and for each succeeding derivative.

Ex. 1. Find $D_x^n y$ when $y = x^2e^x$.

$$D^n y = x^2 \cdot D^n(e^x) + nD(x^2)\cdot D^{n-1}(e^x) + \frac{n(n-1)}{1\cdot 2}D^2(x^2)\cdot D^{n-2}(e^x) + \cdots$$
$$= e^x[x^2 + 2\,nx + n(n-1)].$$

Ex. 2. Calculate the fourth x-derivative of $x^5 \sin x$ by Leibnitz's theorem.

Ex. 3. Find $D_x^n y$ when: (i) $y = xe^x$; (ii) $y = xe^{2x}$.

NOTE. **Reference for collateral reading** *on successive differentiation.* Echols, *Calculus*, Chap. IV., especially Art. 56.

73. Application of differentiation to elimination. It is shown in algebra that one quantity can be eliminated between two independent equations, two quantities between three equations, and that n quantities can be eliminated between $n+1$ independent equations. The process of differentiation can be applied for the elimination of arbitrary constants from a relation involving variables and the constants. For by differentiation a sufficient number of equations can be obtained between which and the original equation the constants can be eliminated.

EXAMPLES.

1. Given that
$$y = A\cos x + B\sin x, \tag{1}$$
in which A and B are arbitrary constants, eliminate A and B.

In order to render possible the elimination of these two constants, two more equations are required. These equations can be obtained by differentiation. Thus,

$$\frac{dy}{dx} = -A\sin x + B\cos x, \tag{2}$$

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x. \tag{3}$$

On eliminating A and B between (1), (2), (3), there is obtained the relation

$$\frac{d^2y}{dx^2} + y = 0. \tag{4}$$

NOTE 1. Equation (4) is called a *differential equation*, as it involves a derivative. It is the differential equation corresponding to, or expressing the same relation as, the "integral" equation (1). The process of deducing the integral equations (or *solutions*, as they are then called) of differential equations is discussed, but for a very few cases only, in Chapter XXI.

2. Eliminate the arbitrary constants m and b from the equation

$$y = mx + b. \qquad \textit{Ans.}\ \frac{d^2y}{dx^2} = 0.$$

In this case the given equation represents all lines, m and b being arbitrary. Accordingly the resulting equation is the differential equation of all lines. For the geometrical point of view see Art. 68, Note 3.

3. Eliminate the arbitrary constants a and b from each of the following equations: (1) $y = ax^2 + b$. (2) $y = ax^2 + bx$. (3) $(y - b)^2 = 4\,ax$. (4) $y^2 - 2\,ay + x^2 = a^2$. (5) $y^2 = b(a^2 - x^2)$.

4. Find the differential equations which have the following equations for solutions, c_1 and c_2 being arbitrary constants:

(1) $y = c_1$. (2) $y = c_1x$. (3) $y = c_1x + c_2$. (4) $y = c_1e^x + c_2e^{-x}$.
(5) $y = c_1e^{mx} + c_2e^{-mx}$. (6) $y = c_1 \cos mx + c_2 \sin mx$. (7) $y = c_1 \cos(mx + c_2)$.

5. Obtain the differential equations of all circles of radius r: (1) which have their centres on the x-axis; (2) which have their centres on the y-axis; (3) which have their centres anywhere in the xy-plane.

6. Show that the elimination of n arbitrary constants $c_1, c_2, \cdots, c_n$, from an equation $f(x, y, c_1, c_2, \cdots, c_n) = 0$ gives rise to a differential equation involving the nth derivative of y with respect to x.

NOTE 2. For geometrical explanations relating to differential equations the student is referred to Murray, *Differential Equations*, Chap. I., which may easily be read now. The reading will widen his mathematical outlook at this stage.

CHAPTER VII.

FURTHER ANALYTICAL AND GEOMETRICAL APPLICATIONS.

VARIATION OF FUNCTIONS. SKETCHING OF GRAPHS. MAXIMA AND MINIMA. POINTS OF INFLEXION.

N.B. This chapter may be studied before Chap. V. is entered upon.

74. Increasing and decreasing functions. When x changes continuously from one value to another, any continuous function of x, say $\phi(x)$, in general also changes. The function may either be increasing or decreasing, or alternately increasing and decreasing. By means of the calculus it is easy to find out how the function behaves when x passes through any value on its way from $-\infty$ to $+\infty$.

Let Δx be a positive increment of x, and $\Delta\phi(x)$ be the corresponding increment of $\phi(x)$. If $\phi(x)$ continually increases when x is changing from x to $x + \Delta x$, then $\Delta\phi(x)$ is positive; and accordingly, $\dfrac{\Delta\phi(x)}{\Delta x}$ is positive. Moreover, this is positive for all positive values of Δx, however small; hence $\lim_{\Delta x \doteq 0} \dfrac{\Delta\phi(x)}{\Delta x}$ is positive, *i.e.* $\phi'(x)$ is positive. In a similar way it can be shown that if $\phi(x)$ continually decreases when x is changing from x to $x + \Delta x$, then $\phi'(x)$ is negative. These facts may be stated thus:

$$\left.\begin{array}{l}\phi'(x) \text{ is positive when } \phi(x) \text{ is increasing, and} \\ \phi'(x) \text{ is negative when } \phi(x) \text{ is decreasing; and conversely.}\end{array}\right\} A.$$

These facts will also be apparent on an inspection of the accompanying graphs.

Let $\phi(x)$ be graphically represented by the curve $ABCDE$, whose equation is

$$y = \phi(x).$$

At any point on this curve, $\dfrac{dy}{dx} = \phi'(x)$.

By Art. 24, the slope of the curve represents the derivative of the function. Now at A, D, and E, the slope is negative, and the ordinate y (the function) is evidently decreasing as x is passing in the positive direction through the values of x at A, D, and E. On the other hand, at B, C, and F, the slope is positive, and the ordinate y is evidently increasing as x is passing in the positive

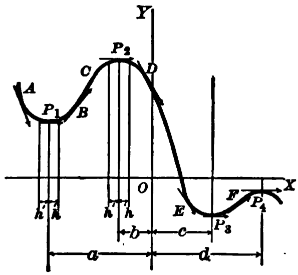

FIG. 30 *a*.

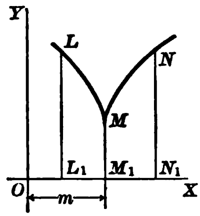

FIG. 30 *b*.

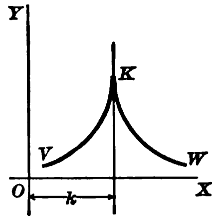

FIG. 30 *c*.

direction through the values of x at B, C, and F. In Fig. 30 *b* when x is increasing from OL_1 to OM_1, the ordinate y is decreasing from L_1L to M_1M and the slope at points on LM is negative; when x is increasing from OM_1 to ON_1, the ordinate is increasing from M_1M to N_1N and the slope at points on MN is positive. Fig. 30 *c* also exemplifies principles marked A on page 116.

75. Maximum and minimum values of a function. Critical points on the graph, and critical values of the variable. The values of the function at points such as P_1, P_2, P_3, M, and K (Art. 74), where the function stops increasing and begins to decrease, or *vice versa*, may be called *turning values* of the function. When a function ceases to increase and begins to decrease, as at P_2, P_4, and K, it is said to have a *maximum* value; when a function ceases to decrease and begins to increase, as at P_1, P_3, and M, it is said to have a *minimum* value. Therefore, at a point (on the graph) where the function has a maximum value the slope changes from positive to negative; at a point where the function has a minimum value the slope changes from negative to positive. (Examine Fig. 30.)

Accordingly, at each of these points the slope (*i.e.* the derivative of the function) is *generally* (see Note 3) either zero or infinitely great.

It should be observed that, although the derivative of a function may be either zero or infinitely great for values of the variable for which the function has a maximum or a minimum value, yet the converse is not always the case. The function may not have a maximum or minimum value when its derivative is zero or infinity.

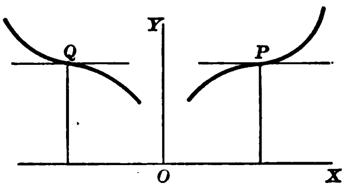

FIG. 31 *a*.

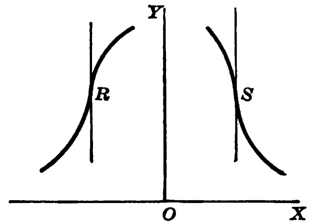

FIG. 31 *b*.

This is exemplified by the functions whose graphs are given in Figs. 31 *a*, *b*. Thus at P the slope is zero and the function is increasing on each side of P; at Q the slope is zero and the function is decreasing on each side of Q; at R the slope is infinitely great, and the function is increasing on each side of R; at S the slope is infinitely great and the function is decreasing on each side of S.

Accordingly, a point where the slope of the graph of a function is zero or infinitely great is, for the purpose of this chapter, called *a critical point.* Such a point must be further criticised, or examined, in order to determine whether the ordinate has either a maximum or a minimum value there. In other words, that value of the variable for which the derivative of a function is zero or infinitely great is called *a critical value;* further examination is necessary in order to determine whether the function is a maximum or a minimum for that value of the variable.

NOTE 1. The points Q, P, R, S (Figs. 31 *a*, *b*), are examples of what are called *points of inflexion* (see Art. 78).

NOTE 2. By saying that a function $\phi(x)$ has a minimum value, for $x = a$ say, it is not meant that $\phi(a)$ is the least possible value the function can have. It is meant that the value of the function for $x = a$ is less than the values of the function for values of x which are on opposite sides of a,

and as close as one pleases to a; *i.e.* h being taken as small as one pleases, $\phi(a) < \phi(a-h)$ and $\phi(a) < \phi(a+h)$. (See P_1 in Fig. 30 *a*.) Likewise, if $\phi(x)$ is a maximum for $x = b$, this means merely that $\phi(b) > \phi(b-h)$ and $\phi(b) > \phi(b+h)$, in which h is as small as one pleases. (See P_2 in Fig. 30 *a*.)

EXAMPLES.

1. Examine $\sin x$ for critical values of the variable.

Here $$\phi(x) = \sin x.$$

The graph of this function is on page 409. In order to find the critical points solve the equation $$\phi'(x) = \cos x = 0.$$

Accordingly, the critical values of x are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$.

2. Examine $(x-1)^2(x+3)$ for critical values of the variable.

Here $$\phi(x) = (x-1)^2(x+3).$$

The solution of $\phi'(x) = (x-1)(3x+5) = 0$,

gives the critical values of x, viz. $1, -\frac{5}{3}$.

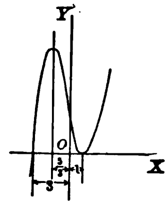

FIG. 31 *c*.

3. Examine $(x-1)^3+2$ for critical values of the variable.

Here $$\phi(x) = (x-1)^3 + 2.$$

On solving $$\phi'(x) = 3(x-1)^2 = 0,$$

the critical value of x is obtained, viz. $x = 1$.

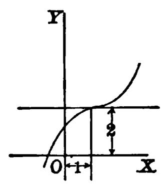

FIG. 31 *d*.

4. Examine $(x-2)^{\frac{2}{3}} + 3$ for critical values of x.

Here $$\phi(x) = (x-2)^{\frac{2}{3}} + 3.$$

On solving $$\phi'(x) = \frac{2}{3(x-2)^{\frac{1}{3}}} = \infty,$$

the critical value $x = 2$ is obtained.

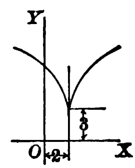

FIG. 31 *e*.

5. Examine $(x-2)^{\frac{1}{3}} + 3$ for critical values of x.

Here $$\phi(x) = (x-2)^{\frac{1}{3}} + 3,$$

and $$\phi'(x) = \frac{1}{3(x-2)^{\frac{2}{3}}} = 0$$

gives the critical value $x = 2$.

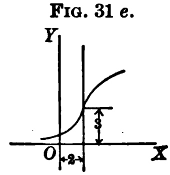

FIG. 31 *f*.

Note 3. A function may have a maximum or minimum value when its derivative changes abruptly; see Art. 164, Note 3, and Fig. 21 (*c*).

76. Inspection of the critical values of the variable (or critical points of the graph) for maximum or minimum values of the function. Let the function be $\phi(x)$. The equation of its graph is $y = \phi(x)$, and the slope is $\frac{dy}{dx}$ or $\phi'(x)$. The solutions of the equations

$$\phi'(x) = 0 \text{ and } \phi'(x) = \infty,$$

give the critical values of the variable.

Suppose that $ABCDE$ (Fig. 30 a) is the graph, and that the critical values are $x = a$ and $x = b$. There are **three ways** of testing whether the critical values of the variable will give maximum or minimum values of the function, viz.:

(a) By examining the function itself at, and on each side of, the critical value;

(b) By examining the first derivative on each side of the critical value;

(c) By examining the second derivative (see Art. 68) at the critical value.

NOTE 1. It follows from the definition of maximum and minimum values, and Note 2, Art. 75, that if $\phi(a)$ is a maximum (or minimum) value of $\phi(x)$, then $\phi(a) + m$, $c\phi(a)$, $\sqrt{\phi(a)}$, $\phi^2(a)$, $\cdots$, are maximum (or minimum) values of $\phi(x) + m$, $c\phi(x)$, $\sqrt{\phi(x)}$, $\phi^2(x)$, $\cdots$, respectively. Accordingly, the finding of critical values of x for one of these functions will give the critical values for the other functions. It sometimes happens that it is much easier to find the critical values for, say $\phi^2(x)$, than for $\phi(x)$. In such a case it is better to examine $\phi^2(x)$ than to examine $\phi(x)$.

(a) Examination of the function. In this test, if $x = a$ is the critical value, $\phi(a - h)$ and $\phi(a + h)$, in which h is as small as one pleases, are both compared with $\phi(a)$. (This is the obvious and natural method of testing the critical values.) *If $\phi(a)$ is greater than both $\phi(a - h)$ and $\phi(a + h)$, then $\phi(a)$ is a maximum; if $\phi(a)$ is less than both $\phi(a - h)$ and $\phi(a + h)$, then $\phi(a)$ is a minimum; if $\phi(a)$ is greater than one and less than the other of $\phi(a - h)$ and $\phi(a + h)$, then $\phi(a)$ is neither a maximum nor a minimum.*

Ex. 1. In Ex. 1, Art. 75, examine the function at the critical value $\frac{\pi}{2}$ of x.

Here $\sin\left(\frac{\pi}{2} - h\right) < \sin\frac{\pi}{2}$, and $\sin\left(\frac{\pi}{2} + h\right) < \sin\frac{\pi}{2}$. Accordingly, $x = \frac{\pi}{2}$ gives a maximum value of $\sin x$.

Ex. 2. (*a*) In Ex. 2, Art. 75, examine the function at the critical value $x = 1$. Here $\phi(1) = 0$, $\phi(1-h) = h^2(4-h)$, $\phi(1+h) = h^2(4+h)$. Accordingly, $\phi(1-h) > \phi(1)$, and $\phi(1+h) > \phi(1)$. Thus $\phi(1)$ is a minimum value of $\phi(x)$.

(*b*) Inspect this function at the critical value $x = -\frac{5}{3}$.

Ex. 3. In Ex. 3, Art. 75, examine the function at the critical value $x = 1$. Here $\phi(1) = 2$, $\phi(1-h) = -h^3 + 2$, and $\phi(1+h) = h^3 + 2$. Accordingly, $\phi(1-h) < \phi(1) < \phi(1+h)$, and thus $\phi(1)$ is not a turning value of the function.

Ex. 4. Examine the functions in Exs. 4, 5, Art. 75, at the critical values of x.

(*b*) Examination of the first derivative of the function. When a function is increasing, its derivative is positive and the slope of its graph is positive; when a function is decreasing, its derivative is negative and the slope of its graph is negative (Art. 74). Hence, h being taken as small as one pleases, *if $\phi'(a-h)$ is positive and $\phi'(a+h)$ is negative, then $\phi(a)$ is a maximum value of $\phi(x)$.* On the other hand, *if $\phi'(a-h)$ is negative and $\phi'(a+h)$ is positive,* then $\phi(x)$ is decreasing when x is approaching a, and $\phi(x)$ is increasing when x is leaving a, and accordingly *$\phi(a)$ is a minimum value of $\phi(x)$.*

NOTE 2. Test (*b*) is generally easier to apply than test (*a*). For test (*a*) the functions $\phi(a-h)$ and $\phi(a+h)$ must be computed; for test (*b*) merely the algebraic signs of $\phi'(a-h)$ and $\phi'(a+h)$ are required.

Ex. 5. (*a*) In Ex. 1, Art. 75. $\phi'\left(\frac{\pi}{2} - h\right)$ is positive and $\phi'\left(\frac{\pi}{2} + h\right)$ is negative. Accordingly, $\phi\left(\frac{\pi}{2}\right)$, *i.e.* $\sin\frac{\pi}{2}$ or 1, is a maximum value of $\sin x$.

(*b*) Apply this test at the other critical values in Ex. 1, Art. 75.

Ex. 6. (*a*) In Ex. 2, Art. 75, $\phi'(1-h)$ is negative and $\phi'(1+h)$ is positive. Accordingly $\phi(1)$, *i.e.* 0, is a minimum value of $(x-1)^2(x+3)$.

(*b*) Apply this test at the other critical value in Ex. 2, Art. 75.

Ex. 7. In Ex. 3, Art. 75, $\phi'(1-h)$ is positive and $\phi'(1+h)$ is positive. Accordingly, $\phi(1)$, or 2, is neither a maximum nor a minimum.

Ex. 8. Apply test (*b*) at the critical values of the functions in Exs. 4, 5, Art. 75.

(*c*) Examination of the second derivative of the function. It has been seen that the sign of the derivative of a function $\phi(x)$ changes from positive to negative when the function is passing through a

maximum value. If the derivative $\phi'(x)$ passes from a positive value to zero, and then becomes negative, the derivative is continually decreasing, and hence *its* derivative, namely $\phi''(x)$, must be negative for the critical value of x. On the other hand, when the function passes through a minimum value, the derivative changes sign from negative to positive. If then the derivative $\phi'(x)$ passes through zero, it is continually increasing, and hence its derivative, namely $\phi''(x)$, must be positive for the critical value of x. Therefore,

if $\phi'(a)$ is zero and $\phi''(a)$ is negative, $\phi(a)$ is a maximum value of $\phi(x)$;

if $\phi'(a)$ is zero and $\phi''(a)$ is positive, $\phi(a)$ is a minimum value of $\phi(x)$.

NOTE 3. When the second derivative can be obtained readily, test (*c*) is the easiest of the three tests to apply.

NOTE 4. Sometimes $\phi''(a)$ is also zero. A procedure to be adopted in this case is discussed in Art. 181. One of the other tests, however, may be used.

NOTE 5. *Historical.* Kepler (1571–1630), the great astronomer, "was the first to observe that the increment of a variable — the ordinate of a curve, for example — is evanescent for values infinitely near a maximum or minimum value of the variable." Pierre de Fermat (1601–1665), a celebrated French mathematician, in 1629 found the values of the variable that make an expression a maximum or a minimum by a method which was practically the calculus method (Art. 75).

NOTE 6. Many problems in maxima and minima may be solved by elementary algebra and trigonometry. For the algebraic treatment see (among other works) Chrystal, *Algebra*, Part II., Chap. XXIV.; Williamson, *Diff. Cal.*, Arts. 133–137; Gibson, *Calculus*, § 76; Lamb, *Calculus*, Art. 52.

NOTE 7. **Maxima and minima of functions of two or more independent variables.** For discussions of this topic see McMahon and Snyder, *Diff. Cal.*, Chap. X., pages 183–197; Lamb, *Calculus*, pages 135, 596–598; Gibson, *Calculus*, §§ 159, 160; Echols, *Calculus*, Chap. XXX.; and the treatises of Todhunter and Williamson.

EXAMPLES.

9. (*a*) In Ex. 1, Art. 75, $\phi''(x) = -\sin x$. Accordingly, $\phi''\left(\frac{\pi}{2}\right)$ is negative, and thus $\phi\left(\frac{\pi}{2}\right)$, *i.e.* $\sin\frac{\pi}{2}$, is a maximum value of $\phi(x)$.

(*b*) Apply test (*c*) at the other critical values of $\sin x$.

10. (*a*) In Ex. 2, Art. 75, $\phi''(x) = 2(3x+1)$. Accordingly, $\phi''(1)$ is positive, and thus $\phi(1)$ is a minimum value of $\phi(x)$.

(*b*) Apply test (*c*) at the other critical value in Ex. 2, Art. 75.

11. In Ex. 3, Art. 75, $\phi''(x) = 6(x-1)$. Here $\phi''(1) = 0$, and thus test (*c*) fails to indicate whether $\phi(1)$ is a turning value of $\phi(x)$. (See Note 4.)

12. Apply test (*c*) at the critical values of the functions in Exs. 4, 5, Art. 75.

Note 8. **Sketching of graphs.** The ideas discussed in Arts. 74–76 are a great aid in making graphs of functions, and in showing what is termed *the march of a function.*

13. For each of the following functions find the critical values of x, determine the maximum and minimum values, and sketch the graphs: (1) $2x^3+5x^2-4x+2$; (2) $5+12x-x^2-2x^3$; (3) $x^2(x+1)(x-2)^3$; (4) $(x-2)^3(x+1)^2$; (5) $2+3(x-4)^{\frac{2}{3}}+(x-4)^{\frac{4}{3}}$; (6) $3x^5-125x^3+2160x$; (7) $\dfrac{x^2-7x+6}{x-10}$; (8) $\dfrac{(x-2)^3}{(x+2)^2}$; (9) $x\log x$; (10) $x^{\frac{1}{x}}$; (11) $2\sin^2 x+8\cos^2 x$; (12) $\sin x \sin 2x$; (13) $x\cos x$.

14. Show that $a+(x-c)^n$ is a minimum when $x=c$, if n is even; and that it has neither a maximum nor a minimum value, if n is odd.

15. (*a*) Show that $(ac-b^2) \div a$ is a maximum or a minimum value of ax^2+bx+c, according as a is positive or negative. (*b*) Show that ax^2+bx+c cannot have both a maximum and minimum value for any values of a, b, c.

16. Find the point of maximum on the curve $x^3+y^3-3axy=0$. Sketch the graph, taking $a=1$.

17. In the case of the ellipse $ax^2+2hxy+by^2+c=0$, show how to find the highest and lowest points, and the points at the extreme right and left.

77. Practical problems in maxima and minima. Some practical applications of the principles of Arts. 75 and 76 will now be given. In making these applications the student is in a position analogous to his position in algebra when he applied his knowledge about the solution of equations to solving "word problems." Here, as in algebra, the most difficult part of the work is the mathematical statement of the problem and the preparation of the *data* for the application of the processes of Art. 76.

EXAMPLES.

1. Find the area of the largest rectangle that can be inserted in a given triangle, when a side of the rectangle lies on a side of the triangle.

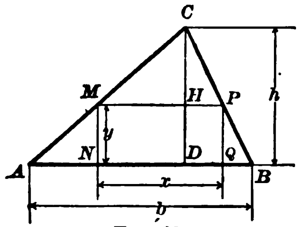

FIG. 32.

Let ABC be the given triangle, and let the given values of the base AB and the height CD be b and h respectively.

Suppose that MQ is the largest rectangle, and let MN and NQ be denoted by y and x respectively, and denote the area of MQ by u.

Then $u = xy$, which is to be a maximum.

It is first necessary to express u, the quantity to be "maximised," in terms of a single variable.

Now $\qquad MP : AB = CH : CD$; *i.e.* $x : b = h - y : h$.

$$\therefore\ x = \frac{b}{h}(h - y);\text{ accordingly, } u = \frac{b}{h}y(h - y),\text{ a maximum.}$$

$\therefore\ \dfrac{du}{dy} = \dfrac{b}{h}(h - 2y) = 0$; whence $y = \frac{1}{2}h$. Thus $x = \frac{1}{2}b$, and area $MQ = \frac{1}{4}bh =$ one half the area of the triangle.

NOTE 1. If M be supposed to move along AC from A to C, the rectangle MQ increases from zero at A and finally decreases to zero at C. It is thus evident that for some point between A and C the rectangle has a maximum value.

NOTE 2. *In these examples it is necessary that the quantity to be maximised or minimised be expressed in terms of one variable.* Conditions sufficient for this must be provided.

2. Solve Ex. 1, expressing u in terms of x.

3. A parabola $y^2 = 8x$ is revolved about the x-axis; find the volume of the largest cylinder that can be inscribed in the paraboloid thus generated, the height of the paraboloid being 4 units.

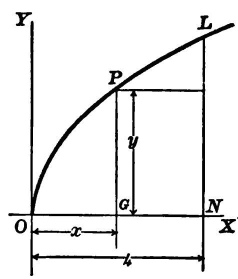

FIG. 33.

Let OPL be the arc that revolves, LN be at right angles to OX, and $ON = 4$. Take $P(x, y)$, a point in OL, and construct the rectangle PN. When OPL generates the paraboloid, PN generates a cylinder. (As P moves along the curve from O to L, the cylinder increases from zero at O and finally decreases to zero at L. Thus there is evidently some position of P between O and L for which the cylinder is a maximum.) Suppose

that PN generates the maximum cylinder, and denote its volume by V. Then

$$V = \pi \overline{PG}^2 \cdot GN = \pi y^2(4 - x) = 8\pi x(4 - x).$$

Accordingly, $\frac{dV}{dx} = 8\pi(4 - 2x) = 0.$

From this, $x = 2$; hence $V = 100.53$ cubic units.

NOTE 3. In the process of maximising in Exs. 1, 2, the constant factors $\frac{b}{h}$ and 8π may as well be dropped. (See Art. 76, Note 1.)

NOTE 4. In each of these examples it is well to perceive at the outset that a maximum or a minimum exists.

4. A man in a boat 6 miles from shore wishes to reach a village that is 14 miles distant along the shore from the point nearest to him. He can walk 4 miles an hour and row 3 miles an hour. Where should he land in order to reach the village in the shortest possible time? Calculate this time. Let L be the position of the boat, M the village, and N the nearest land to L. Then LN is at right angles to NM. Let P denote the place to land, and T denote the time (in hours) to go over $LP + PM$, and denote NP by x.

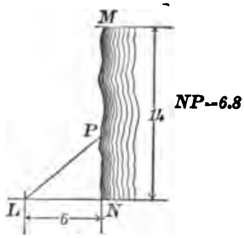

FIG. 34.

Then $$T = \frac{LP}{3} + \frac{PM}{4} = \frac{\sqrt{36 + x^2}}{3} + \frac{14 - x}{4}, \text{ a minimum.}$$

$$\therefore \frac{dT}{dx} = \frac{x}{3\sqrt{36 + x^2}} - \frac{1}{4} = 0.$$

Hence, $x = 6.8$ miles, and $T = 4.8 \cdots$ hours.

5. What must be the ratio of the height of a Norman window of given perimeter to the width in order that the greatest possible amount of light may be admitted? (A Norman window consists of a rectangle surmounted by a semicircle.)

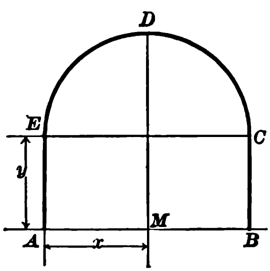

FIG. 35.

Let m denote the given perimeter, $2x$ the width, and y the height of the rectangle in the window desired; let A denote the area of the window.

Then $A = 2xy + \frac{1}{2}\pi x^2.$

Now $2x + 2y + \pi x = m.$

$$\therefore A = mx - 2x^2 - \tfrac{1}{2}\pi x^2,$$

which is to be a maximum.

On finding the value of x for which A is a maximum, and then getting the corresponding value of y, it will appear that $x = y$. Accordingly, the height MD = the width AB.

6. Find the area of the largest rectangle that can be inscribed in an ellipse. (First show that there evidently is such a rectangle.)

SUGGESTIONS: Let the semiaxes of the ellipse be a and b, and choose axes of coördinates coincident with the axes of the ellipse. Let $P(x, y)$ be a vertex of the rectangle. Then area rectangle $= 4xy = 4\frac{b}{a}x\sqrt{a^2 - x^2}$. Maximise the last expression, or, better still, because it is easier to do, maximise the square of $x\sqrt{a^2 - x^2}$, viz. $x^2(a^2 - x^2)$. (See Art. 76, Note 1.) It will be found that the area of the rectangle is $2ab$, half the area of the rectangle circumscribing the ellipse.

7. Divide a number into two factors such that the sum of their squares shall be as small as possible.

8. Two sides of a triangle are given: find, by the calculus, the angle between them such that the area shall be as great as possible.

9. Find the largest rectangle that can be inscribed in a given circle.

10. Through a given point $P(a, b)$ a line is drawn meeting the axes in A and B; O is the origin: Find (i) the least length that AB can have; (ii) the least value of $OA + OB$; (iii) the least possible area of the triangle OAB.

11. A and B are points on the same side of a straight line MN: determine the position of a point C in MN: (1) so that $\overline{AC}^2 + \overline{CB}^2 =$ a minimum; (2) so that $AC + CB =$ a minimum.

N.B. The cones and cylinders in the following examples are *right circular*:

12. (i) Find the height of the cone of greatest volume that can be inscribed in a sphere of radius r. (ii) Find the cone of greatest convex surface that can be inscribed in this sphere.

13. Find the semi-vertical angle of the cone of least volume that can be described about a sphere.

14. (i) Find the cylinder of greatest volume that can be inscribed in a sphere of radius r. (ii) Find the cylinder of greatest curved surface that can be inscribed in this sphere.

15. (i) Determine the maximum cylinder that can be inscribed in a right circular cone of height b and radius of base a. (ii) Determine the cylinder of greatest convex surface that can be inscribed in this cone.

16. What is the ratio of the height to the radius of an open cylindrical can of given volume, when its surface is a minimum?

17. A circular sector of given perimeter has the greatest area possible: find the angle of the sector.

18. It is required to construct from two circular iron plates of radius a a buoy, composed of two equal cones having a common base, which shall have the greatest possible volume: find the radius of the base.

19. An open tank of assigned volume has a square base and vertical sides: if the inner surface is the least possible, what is the ratio of the depth to the width?

20. From a given circular sheet of metal it is required to cut out a sector so that the remainder can be formed into a conical vessel of maximum capacity: show that the angle of the sector removed must be about 66°.

21. In a submarine telegraph cable the speed of signalling varies as $x^2 \log \frac{1}{x}$, where x is the ratio of the radius of the core to that of the covering: show that the speed is greatest when the radius of the covering is $\sqrt{e}$ times the radius of the core.

22. Assuming that the power required to propel a steamer through still water varies as the cube of the speed, find the most economical rate of steaming against a current which is running at a given rate.

23. Assuming that the strength of a rectangular beam varies as the product of the breadth and the square of the depth of its cross-section, find the breadth and depth of the strongest rectangular beam that can be cut from a cylindrical log, the diameter of whose cross-section is d inches.

24. Find the length of the shortest beam that can be used to brace a vertical wall, if the beam must pass over another wall that is a feet high and distant b feet from the first wall.

25. At what distance above the centre of a circle of radius a must an electric light be placed in order that the brightness at the circumference of the circle may be the greatest possible? (Assume that the brightness of a small surface A varies inversely as the square of the distance r from a source of light, and directly as the cosine of the angle between r and the normal to the surface at A.) (Gibson's *Calculus*.)

78. Points of inflexion: rectangular coördinates. As a point moves along the curve LAM from L to M, the tangent at the moving point changes from the position shown at L to that at A

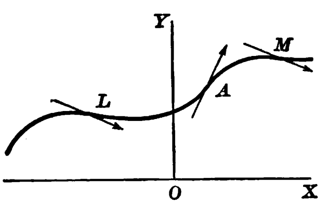

FIG. 36 *a*.

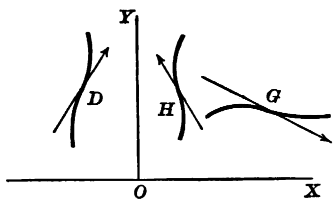

FIG. 36 *b*.

and then to that at M. In going from the position at L to the position at A, the tangent turns in the direction opposite to that in which the hands of a watch revolve; in going from the position

at A to the position of M, the tangent turns in the same direction as that in which the hands of a watch revolve. Points such as A, D, H, G (Fig. 36), and Q, P, R, S (Figs. 31 a, b), at which the tangent for the point moving along the curve ceases to turn in one direction and begins to turn in the opposite direction, are called *points of inflexion.*

Examination of the curve for points of inflexion. As the moving point goes along the curve from L to A, $\frac{dy}{dx}$ increases and accordingly its derivative $\frac{d^2y}{dx^2}$ is positive; as the moving point goes along the curve from A to M, $\frac{dy}{dx}$ decreases, and accordingly $\frac{d^2y}{dx^2}$ is negative. Thus in the case of the curve LAM, $\frac{d^2y}{dx^2}$ is positive on one side of A and negative on the other. Now $\frac{dy}{dx}$ changes continuously from L to M; accordingly, at A $\frac{d^2y}{dx^2}=0.$ Hence, in order to find the points of inflection for a curve $y=f(x)$, proceed as follows:

Calculate $$\frac{d^2y}{dx^2};$$

then solve the equation $$\frac{d^2y}{dx^2}=0.$$

This will give critical values (or points) which are to be further examined or tested. A critical point is tested by finding whether $\frac{d^2y}{dx^2}$ has opposite signs on each side of the point. *If* $\frac{d^2y}{dx^2}$ *has opposite signs, the critical point is a point of inflexion;* if $\frac{d^2y}{dx^2}$ has the same sign on both sides of the critical point, as in Fig. 36 c, the point is what is called *a point of undulation.*

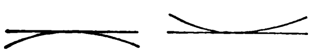

FIG. 36 c.

NOTE 1. At a point of inflexion the tangent crosses the curve. The tangent at an ordinary point on a curve is the limiting position of a secant when two of the points of intersection of the secant and the curve become coincident (Art. 24). The tangent at a point of inflexion is the limiting position of a secant which cuts the curve in more than two points, when the secant revolves until three points of intersection become coincident.

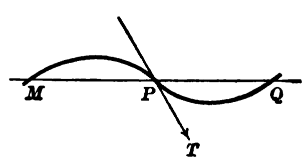

FIG. 37.

Thus PT, the tangent at the point of inflexion P, is the limiting position of the secant MPQ when MPQ revolves about P until M and Q simultaneously coincide with P. At a point of undulation the tangent does not cross the curve. The tangent at a point of inflexion is called *an inflectional tangent;* the tangent where $y'' = 0$ is called *a stationary tangent.*

NOTE 2. If $f(x)$ is a rational integral function of degree n, the greatest number of points of inflexion that the curve $y = f(x)$ can have is $n - 2$. Moreover the points of inflexion occur between points of maxima and minima. [See F. G. Taylor's *Calculus* (Longmans, Green & Co.), Art. 206.]

NOTE 3. **References for collateral reading.** *On maxima and minima of functions of one variable*, etc.: McMahon and Snyder, *Diff. Cal.*, Chap. VI.; Echols, *Calculus*, Chap. VIII. (in particular, Art. 85). *On points of inflexion:* Williamson, *Diff. Cal.* (7th ed.), Arts. 221–224; Edwards, *Treatise on Diff. Cal.*, Arts. 274–279; Echols, *Calculus*, Chap. XI.

NOTE 4. **Points of inflexion: polar coördinates.** For a discussion of this topic see Todhunter, *Diff. Cal.*, Art. 294; Williamson, *Diff. Cal.*, Art. 242; F. G. Taylor, *Calculus*, Art. 276.

EXAMPLES.

1. In the following curves find the points of inflexion, and write the equations of the inflexional tangents; also sketch the curves and draw the inflexional tangents: (1) $y = x^3$; (2) $x - 3 = (y + 3)^3$; (3) $y = x^2(4 - x)$; (4) $12y = x^3 - 6x^2 + 48$; (5) $y = \dfrac{8}{x^2 + 4}$; (6) $y = \dfrac{2x}{1 + x^2}$; (7) $y = \dfrac{x^3}{4 + x^2}$.

2. Find the points of inflexion on the following curves: (1) $y = x(x - a)^4$; (2) $xy^2 = a^2(a - x)$; (3) $ax^2 - x^2y - a^2y = 0$; (4) $y = b + (c - x)^3$; (5) $y = m - b(x - c)^{\frac{2}{3}}$; (6) $x^3 - 3bx^2 + a^2y = 0$.

3. Show that the curve $y = x^4$ has no point of inflexion. Sketch the curve.

4. Show that the points where the curve $y = b \sin \dfrac{x}{a}$ meets the x-axis are all points of inflexion.

5. Show that the curve $(1 + x)^2y = 1 - x$ has three points of inflexion, and that they lie in a straight line.

6. Show why a conic section cannot have a point of inflexion.

7. Show, both geometrically and analytically, why points of inflexion may be called *points of maximum or minimum slope.*

CHAPTER VIII.

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES.

N.B. This chapter may be studied immediately after Chapter VII., or its study may be postponed and taken up after any one of Chapters IX.–XX.*

79. Partial derivatives. Notation. Thus far functions of one independent variable have been treated; functions of two and of more than two independent variables will now be considered.

Let
$$u = f(x, y) \tag{1}$$
in which $f(x, y)$ is a continuous function (see Note 2) of two independent variables x and y. The value of the function for a pair of values of x and y is obtained by substituting these values in $f(x, y)$.

Thus, if $f(x, y) = 3x - 2y + 7$, $f(1, 2) = 3 \cdot 1 - 2 \cdot 2 + 7 = 6$.

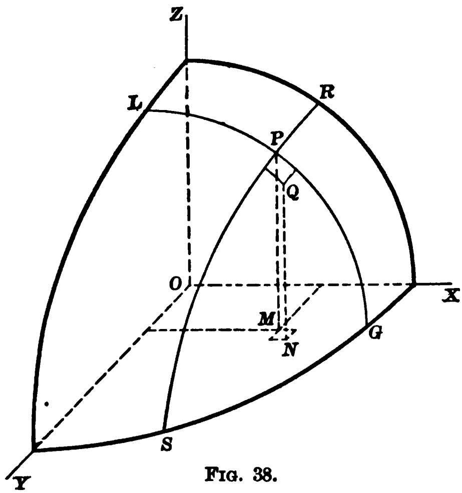

FIG. 38.

NOTE 1. **Geometrical representation of a function of two variables.** The student knows how a continuous function of one variable can be represented by a curve. A continuous function of two variables can be represented by a *surface*. Thus the function z, when
$$z = f(x, y), \tag{2}$$
is represented by the surface $LRGS$ if MP, the perpendicular to the xy-plane erected at any point $M(x, y)$ on that plane and drawn to meet the surface at P, is equal to $f(x, y)$.

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* See the order of the topics in Echols' *Calculus*.

References for collateral reading. See chapters on the geometry of three dimensions in text-books on Analytic Geometry, for instance, those of Tanner and Allen, Ashton, Wentworth; also Echols' *Calculus*, Chap. XXIV.

NOTE 2. **Continuous function of two variables defined.** A function $f(x, y)$ is said to be *a continuous function of x and y* within a certain range of values of x and y, when: (i) $f(x, y)$ does not become infinitely great, and (ii) if, (a, b) and $(a+h, b+k)$ being any values of (x, y) within this range, $f(a+h, b+k)$ can be made to approach as nearly as one pleases to $f(a, b)$ by diminishing h and k, and if $f(a+h, b+k)$ becomes equal to $f(a, b)$, no matter in what way h and k approach to, and become equal to, zero. This definition may be illustrated geometrically, thus: On the xy-plane (Fig. 38) let M be (a, b) and N be $(a+h, b+k)$, and let MP be $f(a, b)$ and NQ be $f(a+h, b+k)$. Then, if MP and NQ are finite, and if NQ remains finite while N approaches M, and becomes equal to MP when N reaches M, no matter by what path of approach on the xy-plane, $f(x, y)$ is said to be a continuous function of x and y for $x = a$ and $y = b$.

In (1) suppose that x receives a change Δx and that y remains unchanged. Then u receives a corresponding change Δu, and

$$u + \Delta u = f(x + \Delta x, y);$$

and
$$\Delta u = f(x + \Delta x, y) - f(x, y).$$

$$\therefore \frac{\Delta u}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

and
$$\lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

This limiting value is called the *partial derivative* of u with respect to x, because there is a like derivative of u with respect to y, namely,
$$\lim_{\Delta y \doteq 0} \frac{\Delta u}{\Delta y} = \lim_{\Delta y \doteq 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

These partial derivatives are usually written

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \tag{3}$$

respectively, in order to distinguish them from derivatives (like $\frac{du}{dx}, \frac{du}{dy}, \frac{ds}{dt}$, and so on) of functions of a single variable and from what are called total derivatives (see Art. 81). If $u = f(x, y, z)$,

the partial derivatives of the first order are $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$. According to the above definition, the partial derivative with respect to each variable is obtained by differentiating the function as if the other variable were constant. Notation (3) is very commonly used, but various other symbols for partial derivatives are also employed.

NOTE 3. **Geometrical representation of partial derivatives of a function of two variables.** Let $f(x, y)$ be represented by the surface $LRGS$ (Fig. 38) whose equation is

$$z = f(x, y).$$

Take P any point (x, y, z) on this surface. Through P pass planes parallel to the planes ZOX and ZOY, and let them intersect the surface in the curves LPG and RPS respectively. Along RPS, x remains constant; and along LPG, y remains constant. Accordingly, from the definition above and Art. 24 the partial x-derivative $\frac{\partial z}{\partial x}$ is the slope of LPG at P, and the partial y-derivative $\frac{\partial z}{\partial y}$ is the slope of RPS at P.

EXAMPLES.

1. If $$u = x^3 + 2x^2y + xy^3 + y^4 + e^x + x\cos y,$$
then $$\frac{\partial u}{\partial x} = 3x^2 + 4xy + y^3 + e^x + \cos y,$$
and $$\frac{\partial u}{\partial y} = 2x^2 + 3xy^2 + 4y^3 - x\sin y.$$

2. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$, when $u = x^3 + 2y^2 + 3z^2 + e^x \sin y + \cos z \cos y$.

3. On the ellipsoid $\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{9} = 1$: (a) find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point where $x = 1$ and $y = 4$; (b) find $\frac{\partial x}{\partial z}$ and $\frac{\partial x}{\partial y}$ at the point where $y = 2$ and $z = 2$; (c) find $\frac{\partial y}{\partial z}$ and $\frac{\partial y}{\partial x}$ at the point where $z = 1$ and $x = 3$. Make figures for (a), (b), and (c), and show what these partial derivatives represent on the ellipsoid.

4. Verify the following:

(i) If $u = \log(e^x + e^y)$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$;

(ii) If $u = \frac{e^{xy}}{e^x + e^y}$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (x + y - 1)u$;

(iii) If $u = x^y y^x$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (x + y + \log u)u$.

80. Successive partial derivatives. The partial derivatives of the first order described in Art. 79 are, in general, also continuous functions of the variables, and *their* partial derivatives may also be required. In the successive differentiation of functions of two or more variables, the following is one of the systems of notation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \text{ is written } \frac{\partial^2 u}{\partial x^2}; \qquad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \text{ is written } \frac{\partial^2 u}{\partial y^2};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \text{ is written } \frac{\partial^2 u}{\partial y\,\partial x}; \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \text{ is written } \frac{\partial^2 u}{\partial x\,\partial y};$$

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial y\,\partial x}\right) \text{ is written } \frac{\partial^3 u}{\partial z\,\partial y\,\partial x}; \qquad \frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x^2}\right) \text{ is written } \frac{\partial^3 u}{\partial z\,\partial x^2};$$

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x\,\partial z}\right) \text{ is written } \frac{\partial^3 u}{\partial z\,\partial x\,\partial z}; \qquad \frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial y^2}\right) \text{ is written } \frac{\partial^3 u}{\partial z\,\partial y^2};$$

and so on.

NOTE 1. In this notation the symbol above the horizontal bar indicates the *order* of the derivative, and the symbols below the bar, *taken from right to left*, indicate the order in which the successive differentiations are to be performed. Thus $\frac{\partial^6 u}{\partial x^2\,\partial y\,\partial z^3}$ means that u is to be differentiated three times in succession with respect to z, and the result is then to be differentiated with respect to y; and the function thus obtained is then to be differentiated twice in succession with respect to x.

NOTE 2. The adoption, by mathematicians, of the symbol ∂ in the notation of partial differentiation was mainly due to the great mathematician, *Carl Gustav Jacob Jacobi* (1804–1851), who decided, in 1841, to use ∂ in the manner which afterwards became the fashion. As to some points of insufficiency and difficulty connected with this notation, see correspondence between Thomas Muir and John Perry, *Nature*, Vol. 66, pages 53, 271, 520.

NOTE 3. **The order in which the successive differentiations are performed does not affect the result** (*certain conditions being satisfied*); *e g.*

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}, \quad \frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial^3 u}{\partial z\,\partial x\,\partial y} = \frac{\partial^3 u}{\partial y\,\partial x\,\partial z}, \quad \frac{\partial^3 u}{\partial z\,\partial x\,\partial z} = \frac{\partial^3 u}{\partial z^2\,\partial x} = \frac{\partial^3 u}{\partial x\,\partial z^2}.$$

This theorem is discussed in Art. 85; it may be verified in the examples that follow, especially in the *important* example, Ex. 1. For a simple example illustrating an unwarranted assumption sometimes made in the proof of this theorem, see Gibson, *Calculus*, page 221.

EXAMPLES.

1. Show that $\dfrac{\partial^2}{\partial x\,\partial y}(Ax^m y^n) = \dfrac{\partial^2}{\partial y\,\partial x}(Ax^m y^n)$, in which A, m, and n are constants. Then show that if $u = \Sigma Ax^m y^n$, $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$, and hence that the theorem in Note 3 is true for all algebraic functions.

2. In the following instances verify the fact that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$; $u = \sin(xy)$; $u = \cos\dfrac{y}{x}$; $u = x^y$; $u = \dfrac{ay - bx}{by - ax}$; $u = \sec(ax + by)$; $u = x\log y$; $u = x\sin y + y\sin x$; $u = y\log(1 + xy)$; $u = \sin(x^y)$; $u = \sin(x)^y$.

3. In the following instances verify that $\dfrac{\partial^3 u}{\partial y^2\,\partial x} = \dfrac{\partial^3 u}{\partial y\,\partial x\,\partial y} = \dfrac{\partial^3 u}{\partial x\,\partial y^2}$: (i) $u = a\tan^{-1}\left(\dfrac{y}{x}\right)$; (ii) $u = \sin(xy) + \dfrac{x + y}{xy}$.

4. Show that $\dfrac{\partial^4 u}{\partial x^2\,\partial y^2} = \dfrac{\partial^4 u}{\partial y^2\,\partial x^2}$, when $u = \cos(ax^n + by^m)$.

5. If $u = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$, show that $\dfrac{\partial^2 u}{\partial x^2} = a^2\dfrac{\partial^2 u}{\partial y^2}$.

6. If $u = \dfrac{x^2y^2}{x + y}$, show that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x\,\partial y} = 2\dfrac{\partial u}{\partial x}$, and that $y\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial^2 u}{\partial x\,\partial y} = 2\dfrac{\partial u}{\partial y}$.

7. If $u = \sqrt[3]{x^2 + y^2}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{2}{9}u$.

8. If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

9. Show that a function of two independent variables has $n + 1$ partial derivatives of order n.

81. Total rate of variation of a function of two or more variables.

N.B. Before reading this article and the next it is advisable to review Arts. 25, 26.

Given that
$$u = f(x, y), \tag{1}$$
and that x and y vary independently of each other, it is required to find the rate of variation of u in terms of the rates of variation of x and y; *i.e.* to find $\dfrac{du}{dt}$ in terms of $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

In (1) let x and y receive increments Δx and Δy respectively, in a time Δt say; then u receives a corresponding increment Δu, and

$$u + \Delta u = f(x + \Delta x, y + \Delta y).$$

$$\therefore\ \Delta u = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{2}$$

Hence, on introduction of $-f(x, y+\Delta y)+f(x, y+\Delta y)$ and division by Δt,

$$\frac{\Delta u}{\Delta t}=\frac{f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)}{\Delta t}+\frac{f(x, y+\Delta y)-f(x, y)}{\Delta t}$$

$$=\frac{f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)}{\Delta x}\cdot\frac{\Delta x}{\Delta t}+\frac{f(x, y+\Delta y)-f(x, y)}{\Delta y}\cdot\frac{\Delta y}{\Delta t}.$$

Now let Δt approach zero; then Δx and Δy approach zero, and, moreover (if a certain condition is satisfied),

$$\lim_{\Delta x \doteq 0\, \Delta y \doteq 0}\frac{f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)}{\Delta x}=\frac{\partial f(x, y)}{\partial x}{}^{*}, \text{ i.e. } \frac{\partial u}{\partial x};$$

and $$\lim_{\Delta y \doteq 0}\frac{f(x, y+\Delta y)-f(x, y)}{\Delta y}=\frac{\partial u}{\partial y}.$$

Hence, $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}. \qquad (3)$$

In words: *The total rate of variation of a function of x and y is equal to the partial x-derivative multiplied by the rate of variation of x plus the partial y-derivative multiplied by the rate of variation of y.*

Similarly, if $$u=f(x, y, z),$$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}. \qquad (4)$$

Results (3) and (4) can be extended to functions of any number of variables. (All derivatives herein are *assumed* to be *continuous.*)

NOTE 1. A function may remain constant while its variables change. The total rate of variation of such a function is evidently zero. (See Art. 84.)

NOTE 2. Suppose that in (1) *y is a function of x* and that the derivative of u with respect to x is required. This may be obtained either directly, as (3) has been obtained, or by substituting x for t in (3); then

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}. \qquad (5)$$

Result (5) may also be obtained by dividing both members of (3) by $\frac{dx}{dt}$ [Art. 34 (3)].

* For a discussion of the condition necessary and sufficient for the passage of the first member of this equation into the second, see W. B. Smith, *Infinitesimal Analysis*, Vol. I, Art. 205 (and also Arts. 206, 207).

NOTE 3. In (5) $\frac{\partial u}{\partial x}$ is the x-derivative of u when y is treated as a constant, and $\frac{du}{dx}$ is the x-derivative of u when y is treated as a function of x.

Here $\frac{du}{dx}$ is called **the total x-derivative of u.**

Similarly the total y-derivative $\frac{du}{dy}=\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x}\frac{dx}{dy}$.

EXAMPLES.

1. Express result (5) in words.

2. Given $$z = 3x^2 + 4y^2, \tag{1}$$
find $\frac{dz}{dt}$ when $x=3$, $y=-4$, $\frac{dx}{dt}=2$ units per second, and $\frac{dy}{dt}=3$ units per second.

On differentiation in (1), $\frac{dz}{dt}=6x\frac{dx}{dt}+8y\frac{dy}{dt}=-60.$

Geometrically this means that on the surface (1), which is an elliptic paraboloid, if a point moves through the point (3, -4, 91) in such a way that the x and y coördinates of the moving point are there increasing at the rates of 2 and 3 units per second respectively, then the z-coördinate of the moving point is, at the same place and moment, decreasing at the rate of 60 units per second.

N.B. Figures should be drawn for Ex. 2 and the following examples.

3. In Ex. 3 (a), Art. 79, find how the z-coördinate is changing when the x-coördinate is increasing at the rate of 1 unit per second, and the y-coördinate is decreasing at the rate of 2 units per second.

4. In Ex. 3 (b), Art. 79, find how x is behaving when y is decreasing at the rate of 2 units per second, and z is increasing at the rate of 3 units per second.

82. Total differential. Let dx and dy be differentials of the x and y in (1) Art. 81. They may be regarded as quantities such that
$$dx : dy = \frac{dx}{dt} : \frac{dy}{dt}.$$
Now let du be taken so that
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \tag{1}$$
As used in (1) $\frac{\partial u}{\partial x}dx$ is called *the partial x-differential of u*, $\frac{\partial u}{\partial y}dy$ is called *the partial y-differential of u*, and du is called the *total differential of u*, and *the complete differential of u*.

NOTE 1. When y is a function of x, relation (1) follows directly from Eq. (5), Art. 81, and definition (5), Art. 27.

NOTE 2. The partial differentials in (1) are also denoted by $d_x u$ and $d_y u$, and thus (1) may be written
$$du = d_x u + d_y u.$$

NOTE 3. In general the du in (1) is not exactly equal to the actual change in u due to the changes dx and dy in x and y; but the smaller dx and dy are taken, the more nearly is du equal to the real change in u (see exercises below). The differential du may be regarded as, and is very useful as, an approximation to the actual change in u. In some cases this change can be calculated directly; in others it can be found to as close an approximation as one pleases by a series developed by means of the calculus. [See Chap. XX., in particular, Art. 176, Eq. (10), and Art. 178, Note 5.]

EXAMPLES.

1. Express relation (1) in words.

2. Given $u = 3x^2 + 2y^2$, find du when $x = 2$, $y = 3$, $dx = .01$, and $dy = .02$.

Here $du = 6x\,dx + 4y\,dy = .12 + .24 = .36$.

The actual change in u is $3(2 \cdot 01)^2 + 2(3 \cdot 02)^2 - (3 \cdot 2^2 + 2 \cdot 3^2) = .3611$.

3. As in Ex. 2 when $dx = .001$ and $dy = .002$. Also find the change in u.

4. Find the complete differential of each of the following functions:

(i) $\tan^{-1}\dfrac{y}{x}$; (ii) y^x; (iii) x^y; (iv) $\log x^y$; (v) $u = x^{\log y}$.

5. Find dy when $y = 8 \cos A \sin B$, $A = 40°$, $dA = 30'$, $B = 65°$, $dB = 20'$.

NOTE 4. It may be said here that if $LRGS$ (Fig. 38) be the surface $z = f(x, y)$, and if M be (x, y) and N be $(x + dx, y + dy)$, and NQ be produced to meet in Q_1 the plane tangent to the surface at P, then the total differential dz is equal to $NQ_1 - MP$.

Ex. Prove this statement. (Suggestion: make a good figure.)

Similarly to (1), if $u = f(x, y, z)$, and dx, dy, dz, be differentials of x, y, z, respectively, and if du be taken so that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz, \tag{2}$$

du is called the *total differential of u*. Relation (2) is also written

$$du = d_x u + d_y u + d_z u.$$

Definitions (1) and (2) may be extended to functions of any number of variables.

6. Given $u = x^2 + y^2 + 2z$, find du when $x = 2$, $y = 3$, $z = 4$, $dx = .1$, $dy = .4$, $dz = -.3$. Also find the actual change in u.

7. The numbers u, x, y, and z being as in Ex. 6, $dx = .01$, $dy = .04$, and $dz = -.03$, calculate the difference between du and the actual change in u.

8. Find du when $u = x^{y^z}$.

83. Approximate value of small errors. A practical application of relations (1) and (2), Art. 82, may be made to the calculation of approximate values of small errors. The ideas set forth in the first part of Art. 65 may be applied to any number of variables.

If $$u = f(x, y, z, \cdots),$$

and dx, dy, dz, $\cdots$, be regarded as errors in the assigned or measured values of x, y, z, $\cdots$, then

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz + \cdots$$

is, *approximately*, the value of the consequent error in the computed value of u. Illustrations can be obtained by adapting Exs. 2, 3, 5, 6, 7, Art. 82. In applying the calculus to the computation of approximate values of errors it is usual to denote the errors (or differences) in $u, x, y, \cdots$, by Δu, Δx, Δy, $\cdots$, rather than by du, dx, dy, $\cdots$. Other notations are also used; *e.g.* δu, δx, δy, $\cdots$.

EXAMPLES.

1. In the cylinder in Ex. 3, Art. 65, give an approximate value of the error in the computed volume due to errors Δh in the height and Δr in the radius.

Let V denote the volume. Then $V = \pi r^2 h$.

$$\therefore \Delta V = 2\pi r h \cdot \Delta r + \pi r^2 \cdot \Delta h.$$

The relative error is $$\frac{\Delta V}{V} = \frac{2\,\Delta r}{r} + \frac{\Delta h}{h}.$$

2. Do as in Ex. 1 for a few concrete cases, and compare the above approximate value of the error with the actual error. What is the difference between the actual error in the volume in Ex. 1 and its approximate value obtained by the method above?

3. In the triangle in Ex. 7, Art. 65, let Δa, Δb, ΔC, be small errors made in the measurement of a, b, C: show that the approximate relative error for the computed area A is $\frac{\Delta a}{a} + \frac{\Delta b}{b} + \cot C \cdot \Delta C$.

Find, by the calculus, an approximate value of ΔA, given that $a = 20$ inches, $b = 35$ inches, $C = 48° 30'$, $\Delta a = .2$ inch, $\Delta b = .1$ inch, $\Delta C = 20'$. How can the actual error in the computed area be obtained?

4. Show that for the area A of an ellipse when small errors are made in the semiaxes a and b, approximately $\frac{\Delta A}{A} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$.

In this general case, and in several concrete cases, compare the approximate error in the computed area with the actual error.

5. In the case described in Ex. 3 show that if Δc denote the consequent error in the computed value of c, then, approximately,

$$\Delta c = \cos B \cdot \Delta a + \cos A \cdot \Delta b + a \sin B \cdot \Delta C.$$

N.B. For remarks and examples on this topic see Lamb, *Calculus*, pp. 138–142, Gibson, *Calculus*, pp. 258–260.

84. Differentiation of implicit functions, two variables. This topic has been taken up in one way in Art. 56. Let the relation connecting two variables x and y be in the implicit form

$$f(x, y) = c, \tag{1}$$

in which c denotes any constant, including zero. Let u denote the function $f(x, y)$; then (1) may be written

$$u = c. \tag{2}$$

Since u remains constant when x and y change, $\frac{du}{dt} = 0$; *i.e.* (Art. 81, Eq. 3, and Note 1)

$$\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = 0. \tag{3}$$

From (3), $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\dfrac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$; whence [Art. 34, Eq. (3)], $\dfrac{dy}{dx} = -\dfrac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$. (4)

Ex. 1. Express relation (4) in words.

NOTE. It should not be forgotten that the relation between the function and the variable should be expressed in form (1) before (4) is applied.

Ex. 2. Do Exs. 13, 14, Art. 37, and exercises, Art. 56, by the method of this article. Compare the methods of Arts. 37, 56, and 84.

85. Order of partial differentiations commutative. The theorem stated in Art. 80, Note 3, and illustrated by the exercises there, will now be proved. (See Gibson, *Calculus*, § 93, especially pages 221, 222.) Suppose that

$$u = f(x, y) \tag{1}$$

and that *u and its first and second partial derivatives are continuous* over a finite range of the variables; then, as will now be shown,

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}. \tag{2}$$

Let x receive an increment h, and y remain constant; then by the theorem of mean value (Art. 64, Eq. 3)

$$f(x+h, y) - f(x, y) = h\frac{\partial}{\partial x} f(x + \theta_1 h, y), \text{ in which } 0 < \theta_1 < 1. \tag{3}$$

Now let y receive an increment k, and x remain constant; then on applying the theorem of mean value to the second member of (3),

$$[f(x+h, y+k) - f(x, y+k)] - [f(x+h, y) - f(x, y)]$$

$$= k\frac{\partial}{\partial y}\left[h\frac{\partial}{\partial x} f(x+\theta_1 h, y+\theta_2 k)\right] = hk\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} f(x+\theta_1 h, y+\theta_2 k)\right], \tag{4}$$

in which $0 < \theta_2 < 1$. On giving the increments in the reverse order,

$$[f(x+h, y+k) - f(x+h, y)] - [f(x, y+k) - f(x, y)]$$

$$= hk\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} f(x+\theta_3 h, y+\theta_4 k)\right], \tag{5}$$

in which θ_3 and θ_4 lie between 0 and 1. Hence, on equating the values of

$$\frac{f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)}{hk}$$

derived from equations (4) and (5),

$$\frac{\partial}{\partial y}\cdot\frac{\partial}{\partial x} f(x+\theta_1 h, y+\theta_2 k) = \frac{\partial}{\partial x}\,\frac{\partial}{\partial y} f(x+\theta_3 h, y+\theta_4 k), \tag{6}$$

provided that $x + h$ and $y + k$ are within the range referred to above. Now let h and k approach zero. Then, since the first and second partial derivatives are continuous, equation (6) becomes

$$\frac{\partial^2 f(x, y)}{\partial y\, \partial x} = \frac{\partial^2 f(x, y)}{\partial x\, \partial y}; \quad i.e. \quad \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}.$$

The proof of the commutation theorem can be extended to derivatives of higher orders and to functions of more than two variables.

NOTE 1. **References for collateral reading** on *partial differentiation, total differentials,* and *the commutative property;* Todhunter, *Diff. Cal.;* McMahon and Snyder, *Diff. Cal.*, Arts. 91–102; Lamb, *Calculus* (ed. 1897), Arts. 45, 46, 60–62, 209, 210; Edwards, *Treatise on Diff. Cal.*, Chap. VI. Especially full and clear treatment of differentiation of functions of more than one variable, with various illustrations and geometrical interpretation, is given in Gibson's *Calculus*, Chap. XI. (see in particular pp. 204–225 and Ex. 1, p. 222), and in Echols' *Calculus*, Chaps. XXV.–XXIX., pp. 282–313.

NOTE 2. **Applications of partial differentiation:** (*a*) To the determination of the maximum and minimum values of functions of two or more variables (see references in Art. 76, Note 7); (*b*) To the study of surfaces, and curves in space (see references, Art. 166, Note 2).

86. Condition that an expression of the form $Pdx + Qdy$ be a total differential. This article may be regarded as supplementary to Art. 82.

Suppose that $f_1(x, y)$ and $f_2(x, y)$ are two arbitrarily chosen functions: does a function exist which has $f_1(x, y)$ for its partial x-derivative and $f_2(x, y)$ for its partial y-derivative? A little thought leads to the conclusion that *in general such a function does not exist.* The condition that must be satisfied in order that there may be such a function will now be found. Suppose that there is such a function, and let it be denoted by u. Then, according to the hypothesis,

$$\frac{\partial u}{\partial x} = f_1(x, y) \quad \text{and} \quad \frac{\partial u}{\partial y} = f_2(x, y). \tag{1}$$

By Art. 85,

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}. \tag{2}$$

Hence, from (1) and (2),

$$\frac{\partial}{\partial y} f_1(x, y) = \frac{\partial}{\partial x} f_2(x, y). \tag{3}$$

Result (3) is directly applicable to the differential expression $Pdx + Qdy$ on substituting P for $f_1(x, y)$ and Q for $f_2(x, y)$.

Otherwise: If $Pdx + Qdy$ is a total differential, du say, then

$$\frac{\partial u}{\partial x} = P \quad \text{and} \quad \frac{\partial u}{\partial y} = Q. \tag{4}$$

Hence, from (2) and (4),

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{5}$$

When condition (5) is satisfied, $Pdx + Qdy$ is also called *an exact differential.*

NOTE 1. That this condition is not only *necessary* (as shown above), but also *sufficient*, is shown in works on Differential Equations. (*E.g.* see Professor McMahon's proof in Murray, *Diff. Eqs.*, Note E.)

NOTE 2. For the condition that an expression of the form $Pdx + Qdy + Rdz$ (see Art. 82, Eq. 2) be a total differential, see works on Differential Equations; *e.g.* Murray, *Diff. Eqs.*, Art. 102 and Art. 103, Note.

Ex. 1. Apply test (5) in the following cases: (*a*) $u = 3x^2 + 2y^2$; (*b*) $u = \tan\frac{y}{x}$; (*c*) $x\,dy + y\,dx$; (*d*) $x\,dy - y\,dx$.

Ex. 2. Illustrate by examples the phrase, "*in general* such a function does not exist," which occurs in this article.

87. Euler's theorem on homogeneous functions. Let u be a homogeneous function of x and y of degree n; *i.e.* let

$$u = Ax^n + \cdots + Bx^py^q + Cx^ry^s + \cdots + My^n,$$

in which

$$p + q = r + s = \cdots = n.$$

Then

$$\frac{\partial u}{\partial x} = nAx^{n-1} + \cdots + pBx^{p-1}y^q + rCx^{r-1}y^s + \cdots;$$

$$\frac{\partial u}{\partial y} = \cdots + qBx^py^{q-1} + sCx^ry^{s-1} + \cdots + nMy^{n-1}.$$

From this, on multiplication and simplification,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \qquad (1)$$

This result can be extended to homogeneous functions of any number of variables; thus,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \cdots = nu. \qquad (2)$$

Result (2) is called Euler's theorem.*

(See Williamson, *Diff. Cal.*, Arts. 102–104, 123; McMahon and Snyder, *Diff. Cal.*, Art. 100; Gibson, *Calculus*, page 412.)

* From Leonhard Euler (1707–1783), an eminent Swiss mathematician, who worked at Berlin and St. Petersburg. He greatly advanced the subjects of algebra, trigonometry, and the calculus.

Ex. 1. Prove theorem (2) when u is a homogeneous function of x, y, z.

Ex. 2. Illustrate (1) and (2) by examples in which n is an integer.

Ex. 3. Verify Euler's theorem in the following cases:

$$\text{(i)}\ u=(x^{\frac{2}{3}}+y^{\frac{2}{3}})(x^4+y^4);\quad \text{(ii)}\ u=(x^{\frac{2}{7}}+y^{\frac{2}{7}})\div(x^{\frac{1}{6}}+y^{\frac{1}{6}});$$

$$\text{(iii)}\ u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}.\quad \text{(Here } n=0.)$$

Ex. 4. Verify Euler's theorem when $u=f\left(\frac{y}{x}\right)$; and apply to $\tan\frac{y}{x}$, $\sin^{-1}\frac{y}{x}$, $\log\frac{y}{x}$, in particular. (In this f-function $n=0$.)

88. Successive total derivatives. An example will be given in order to show the procedure.

If
$$u=f(x, y), \tag{1}$$

then (Art. 81, Eq. 3)
$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{2}$$

On differentiation with respect to t again,

$$\frac{d^2u}{dt^2}=\frac{d}{dt}\left(\frac{du}{dt}\right)=\frac{d}{dt}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{dx}{dt}+\frac{\partial u}{\partial x}\frac{d^2x}{dt^2}+\frac{d}{dt}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{dy}{dt}+\frac{\partial u}{\partial y}\frac{d^2y}{dt^2}. \tag{3}$$

Now, in general, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions of x and y; hence, on applying the principle enunciated in Art. 81,

$$\frac{d}{dt}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{dx}{dt}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{dy}{dt},$$

and
$$\frac{d}{dt}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{dx}{dt}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{dy}{dt}.$$

On substituting these values in (3), using the notation of Art. 80, and remembering that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$, Equation (3) becomes

$$\frac{d^2u}{dt^2}=\frac{\partial^2u}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2u}{\partial x\,dy}\frac{dx}{dt}\frac{dy}{dt}+\frac{\partial^2u}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial u}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial u}{\partial y}\frac{d^2y}{dt^2}. \tag{4}$$

N.B. *Questions and exercises suitable for practice and review on the subject-matter of this chapter will be found at page* 386

CHAPTER IX.

CHANGE OF VARIABLE.

N.B. **If it is thought desirable, the study of this chapter may be postponed until some of the following chapters are read.**

89. Change of variable. It is sometimes advisable to change either, or both, of the variables in a derivative. If the relation between the old and the new variables is known, the given derivative can be expressed in terms of derivatives involving the new variable, or variables. Arts. 91–93 are concerned with showing how this may be done. In Art. 90 an expression for the given derivative is found when the dependent and independent variables are interchanged; in Art. 91, when the dependent variable is changed; in Art. 92, when the independent variable is changed; and in Art. 93, when both the dependent and the independent variables are expressed in terms of a single new variable. In Note 1, Art. 93, an example is worked in which the dependent and the independent variables are both expressed in terms of two new variables.

N.B. *Principle* (2) *of Art.* 34 *is repeatedly employed in Arts.* 90–93.

90. Interchange of the dependent and independent variables. Let y be the dependent and x the independent variable. This article shows how to express the successive derivatives of y with respect to x in terms of the derivatives of x with respect to y.

From the fact that $\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} = 1$, and Art. 20 ($c$), it follows that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Again, $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dy}{dx} \quad \text{(Art. 34)}$$

$$=\frac{d}{dy}\left\{\frac{1}{\frac{dx}{dy}}\right\}\div\frac{dx}{dy}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}.$$

Ex. **Express the third x-derivative of y in terms of y-derivatives of x.**

91. Change of the dependent variable. Let the dependent and independent variables be denoted by y and x respectively. It is required to express the successive derivatives of y with respect to x, in terms of the derivatives of z with respect to x when

$$y=F(z).$$

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=F'(z)\cdot\frac{dz}{dx}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left[F'(z)\frac{dz}{dx}\right]$$

$$=F'(z)\frac{d^2z}{dx^2}+\frac{dz}{dx}\cdot\frac{d}{dx}F'(z)=F'(z)\frac{d^2z}{dx^2}+\frac{dz}{dx}\cdot\frac{d}{dz}[F'(z)]\cdot\frac{dz}{dx}$$

$$=F'(z)\frac{d^2z}{dx^2}+F''(z)\left(\frac{dz}{dx}\right)^2.$$

Ex. Given that $y = F(z)$, show that

$$\frac{d^3y}{dx^3}=F'(z)\frac{d^3z}{dx^3}+3F''(z)\cdot\frac{d^2z}{dx^2}\frac{dz}{dx}+F'''(z)\left(\frac{dz}{dx}\right)^3.$$

92. Change of the independent variable. Let the dependent and independent variables be denoted by y and x respectively. It is required to express the successive derivatives of y with respect to x, in terms of the derivatives of y with respect to z when

$$x=f(z).$$

Here $$\frac{dx}{dz}=f'(z), \text{ and hence, } \frac{dz}{dx}=\frac{1}{f'(z)}.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{f'(z)}\cdot\frac{dy}{dz}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dz}\left(\frac{dy}{dx}\right)\cdot\frac{dz}{dx}=\frac{d}{dz}\left(\frac{1}{f'(z)}\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$=\frac{1}{f'(z)}\left[\frac{1}{f'(z)}\cdot\frac{d^2y}{dz^2}-\frac{f''(z)}{[f'(z)]^2}\cdot\frac{dy}{dz}\right].$$

Ex. Find $\frac{d^3y}{dx^3}$ when $x=f(z)$.

93. Dependent and independent variables both expressed in terms of a single variable.

Let $$y=\phi(t) \text{ and } x=f(t).$$

Then $$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}\,[\text{Art. } 34, (3)]=\frac{\phi'(t)}{f'(t)}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}=\frac{d}{dt}\left[\frac{\phi'(t)}{f'(t)}\right]\cdot\frac{1}{f'(t)}$$

$$=\frac{f'(t)\phi''(t)-\phi'(t)f''(t)}{[f'(t)]^3}. \qquad \text{(Compare Art. 71.)}$$

EXAMPLES.

1. In the above case find $\frac{d^3y}{dx^3}$.

2. Given that $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$, calculate

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\div\frac{d^2y}{dx^2}. \qquad \text{(See Ex. 9, Art. 68.)}$$

3. Given that $x=a\cos\theta$ and $y=a\sin\theta$, calculate the same function as in Ex. 2. What curve is denoted by these equations?

4. Given that $x=a\cos\theta$ and $y=b\sin\theta$, calculate the same function as in Ex. 2. What curve is denoted by these equations?

Note 1. *Both dependent and independent variables expressed in terms of two new variables.* Following is an example of this.

Ex. Given the transformation from rectangular to polar coördinates, viz.

$$x=r\cos\theta,\quad y=r\sin\theta, \qquad (1)$$

express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of r, θ, and the derivatives of r with respect to θ.

From (1), $\frac{dx}{d\theta} = \cos\theta \frac{dr}{d\theta} - r\sin\theta,\ \frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta.$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{d\theta} \div \frac{dx}{d\theta}, \text{ Art. 34, Eq. (3)}\right) = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)^3}.$$

NOTE 2. For more complex cases of change of the variables in a derivative, see other text-books.

NOTE 3. **References for collateral reading.** Williamson, *Diff. Cal.*, Chap. XXII.; McMahon and Snyder, *Diff. Cal.*, Chap. XI.; Edwards, *Treatise on Diff. Cal.*, Chap. XIX.; Gibson, *Calculus*, §§ 98, 99.

EXAMPLES.

N.B. In working these examples it is much better not to use the results or formulas derived in Arts. 90–93, but *to employ the method by which these results have been obtained.*

1. Change the independent variable from x to y in: (i) $\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 = 0$; (ii) $3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0.$

2. In $\frac{d^2y}{dx^2} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2$, change the dependent variable from y to z, given that $y = \tan z$.

3. Change the independent variable under the following conditions: (i) $x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + u = 0$, $y = \log x$; (ii) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$, $x = \cos t$; (iii) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$, $x = \cos t$; (iv) $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \frac{a^2}{x^2}y = 0$, $xz = 1$; (v) $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, $z = \log x$; (vi) $x^4\frac{d^4y}{dx^4} + 6x^3\frac{d^3y}{dx^3} + 9x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \log x$, $x = e^z$.

4. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when: (i) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$; (ii) $x = \cot t$, $y = \sin^3 t$.

5. If $x\frac{d^2y}{dx^2} - \frac{x}{y}\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 0$, and $x = ye^z$, show that $y\frac{d^2z}{dy^2} + \frac{dz}{dy} = 0$.

CHAPTER X.

INTEGRATION.

N.B. If thought desirable, Art. 97 may be studied before Arts. 95, 96. (Remarks relating to the order of study are in the preface.)

94. Integration and integral defined. Notation. In Chapter III. a fundamental process of the calculus, namely, *differentiation*, was explained. In this chapter two other fundamental processes of the calculus, each called *integration*, are discussed. The process of differentiation is used for finding derivatives and differentials of functions; that is, for obtaining from a function, say $F(x)$, its derivative $F'(x)$, and its differential $F'(x)\,dx$. On the other hand the process of integration is used:

(*a*) *For finding the limit of the sum of an infinite number of infinitesimals which are in the differential form* $f(x)\,dx$ (see Art. 96);

(*b*) *For finding functions whose derivatives or differentials are given;* that is, for finding anti-derivatives and anti-differentials (see Arts. 27 *a*, 97).

Briefly, integration may be either (*a*) *a process of summation*, or (*b*) *a process which is the inverse of differentiation*, and which, accordingly, may be called *anti-differentiation*. Integration, as a process of summation, was invented before differentiation. It arose out of the endeavor to calculate plane areas bounded by curves. An area was (supposed to be) divided into infinitesimal strips, and the limit of the sum of these was found. The result was *the whole* (area); accordingly it received the name *integral*, and the process of finding it was called *integration*. In many practical applications integration is used for purposes of summation. In many other practical applications it is not a sum but an anti-differential that is required. It will be seen in Art. 96 that a knowledge of anti-differentiation is exceedingly useful in the process of summation. Exercises on anti-differentiation have appeared in preceding articles.

NOTE. The part of the calculus which deals with differentiation and its immediate applications is usually called *The Differential Calculus*, and the part of the calculus which deals with integration is called *The Integral Calculus*. With **Leibnitz** (1646–1716), the differential calculus originated in the problem of constructing the tangent at any point of a curve whose equation is given. This problem and its inverse, namely, the problem of determining a curve when the slope of its tangent at any point is known, and also the problem of determining the areas of curves, are discussed by Leibnitz in manuscripts written in 1673 and subsequent years. He first published the principles of the calculus, using the notation still employed, in the periodical, *Acta Eruditorum*, at Leipzig in 1684, in a paper entitled *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus.* **Isaac Newton** (1642–1727) was led to the invention of the same calculus by the study of problems in mechanics and in the areas of curves. He gives some description of his method in his correspondence from 1669 to 1672. His treatise, *Methodus fluxionum et serierum infinitarum, cum ejusdem applicatione ad curvarum geometriam*, was written in 1671, but was not published until 1736. The principles of his calculus were first published in 1687 in his *Principia* (*Philosophiae Naturalis Principia Mathematica*). It is now generally agreed that Newton and Leibnitz invented the calculus independently of each other. For an account of the invention of the calculus by Newton and Leibnitz, see Cajori, *History of Mathematics*, pp. 199–236, and Cantor, *Geschichte der Mathematik*, Vol. 3, pp. 150–172.

"There are certain focal points in history toward which the lines of past progress converge, and from which radiate the advances of the future. Such was the age of Newton and Leibnitz in the history of mathematics. During fifty years preceding this era several of the brightest and acutest mathematicians bent the force of their genius in a direction which finally led to the discovery of the infinitesimal calculus by Newton and Leibnitz. Cavalieri, Roberval, Fermat, Descartes, Wallis, and others, had each contributed to the new geometry. So great was the advance made, and so near was their approach toward the invention of the infinitesimal analysis, that both Lagrange and Laplace pronounced their countryman, Fermat, to be the true inventor of it. The differential calculus, therefore, was not so much an individual discovery as the grand result of a succession of discoveries by different minds." (Cajori, *History of Mathematics*, p. 200.)

Also see the "Historical Introduction" in the article, *Infinitesimal Calculus* (*Ency. Brit.*, 9th edition), and, at the end of that article, the list of works bearing on the infinitesimal method before the invention of the calculus.

Notation. In differentiation d and D are used as symbols; thus, $df(x)$ is read "the differential of $f(x)$," and $Df(x)$ is read "the

derivative of $f(x)$." In integration, whether the object be summation or anti-differentiation, the sign $\int$ is most generally used as the symbol; thus, $\int f(x)\,dx$ is read "*the integral of* $f(x)\,dx$."* Other symbols, viz. $d^{-1}f(x)\,dx$ and $D^{-1}f(x)$, are used occasionally (see Art. 97, Note 2). The quantity $f(x)$ which appears "under the integration sign," as the mathematical phrase goes, is called **the integrand.**

95. Examples of the summation of infinitesimals. These examples are given in order to help the student to understand clearly what the phrase "to find the limit of the sum of a set of infinitesimals of the form $f(x)dx$ (*i.e.* a set of infinitesimal differentials)" means.

(*a*) *Find the area between the line* $y = mx$, *the* x-*axis, and the ordinates drawn to the line at* $x = a$ *and* $x = b$.

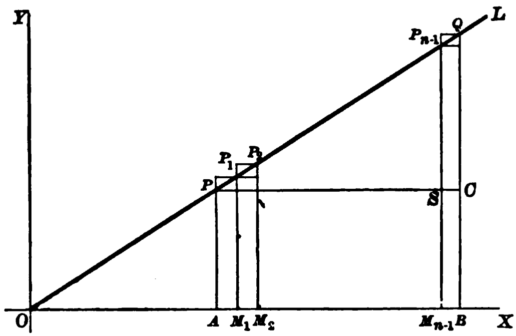

FIG. 39.

Let PQ be the line whose equation is $y = mx$, $OA = a$, and $OB = b$. Draw the ordinates AP and BQ; it is required to find the area $APQB$.

Suppose that AB is divided into n equal parts each equal to Δx, so that

$$n \cdot \Delta x = b - a.$$

Draw the ordinates at each point of division, M_1, M_2, $\cdots$, M_{n-1}; complete the inner rectangles PM_1, P_1, M_2, $\cdots$, $P_{n-1}B$; and complete the outer rectangles P_1A, P_2M_1, $\cdots$, QM_{n-1}. The area $APQB$ is evidently greater than the sum of the inner rectangles and less than the sum of the outer rectangles; *i.e.*

$$\text{sum of inner rectangles} < APQB < \text{sum of outer rectangles}.$$

* The word *integral* appeared first in a solution of James Bernoulli (1654–1705), which was first published in the *Acta Eruditorum* in 1690. Leibnitz had called the integral calculus *calculus summatorius*, but in 1696 the term *calculus integralis* was agreed upon by Leibnitz and John Bernoulli (1667–1748). The sign $\int$ was first used in 1675, and is due to Leibnitz. It is merely the long S which is the initial letter of *summa*, and was used by earlier writers to denote "the sum of,"

The difference between the sum of the inner and the sum of the outer rectangles is the sum of the rectangles PP_1, P_1P_2, $\cdots$, $P^{n-1}Q$. The latter sum is evidently equal to the rectangle QS, *i.e.* to $CQ \cdot \Delta x$. This approaches zero when Δx approaches zero. Therefore $APQB$ is the limit of the sum of either set of rectangles when Δx approaches zero. The limit of the sum of the inner rectangles will now be found.

At A,	$x = a$,	and hence,	$AP = ma$;
at M_1,	$x = a + \Delta x$,	and hence,	$M_1P_1 = m(a + \Delta x)$;
at M_2,	$x = a + 2\,\Delta x$,	and hence,	$M_2P_2 = m(a + 2\,\Delta x)$;
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at M_{n-1},	$x = a + \overline{n-1}\,\Delta x$,	and hence,	$M_{n-1}P_{n-1} = m(a + \overline{n-1} \cdot \Delta x)$,

$\therefore$ sum of inner rectangles

$$= ma \cdot \Delta x + m(a + \Delta x) \cdot \Delta x + m(a + 2\,\Delta x) \cdot \Delta x + \cdots$$
$$+ m(a + \overline{n-1} \cdot \Delta x) \cdot \Delta x.$$

$$\therefore \text{ area } APQB = \lim_{\Delta x \doteq 0} [ma\,\Delta x + m(a+\Delta x)\Delta x + \cdots + m(a + \overline{n-1} \cdot \Delta x)\Delta x]$$
$$= \lim_{\Delta x \doteq 0} m[a + (a+\Delta x) + (a + 2\,\Delta x) + \cdots + (a + \overline{n-1} \cdot \Delta x)]\Delta x.$$

Hence, on summation of the arithmetic series in brackets,

$$\text{area } APQB = \lim_{\Delta x \doteq 0} \frac{mn\,\Delta x}{2}\{2\,a + \overline{n-1} \cdot \Delta x\}.$$

On giving $n\,\Delta x$ its value $b - a$, this becomes

$$\text{area } APQB = \lim_{\Delta x \doteq 0} \frac{m(b-a)}{2}(b + a - \Delta x)$$
$$= m\left(\frac{b^2}{2} - \frac{a^2}{2}\right).$$

NOTE 1. In this example *the element of area*, as it is called, is a rectangle of height y and width Δx when Δx is made infinitesimal, *i.e.* the element of area is $y\,dx$ or $mx\,dx$ in which $dx \doteq 0$. (See Art. 27, Notes 3, 4, and Art. 67 *a*.)

NOTE 2. It may be observed in passing that on taking the anti-differential of $mx\,dx$, namely $\frac{mx^2}{2}$, substituting b and a in turn for x therein, and taking the difference between the results, the required area is obtained.

EX. Find the limit of the sum of the outer rectangles when Δx approaches zero.

(*b*) *Find the area between the parabola $y = x^2$, the x-axis, and the ordinates at $x = a$ and $x = b$.*

Let LOQ be the parabola, $OA = a$, $OB = b$; draw the ordinates AP and BQ; the area $APQB$ is required. As in the preceding problem, divide AB into n parts each equal to Δx, so that

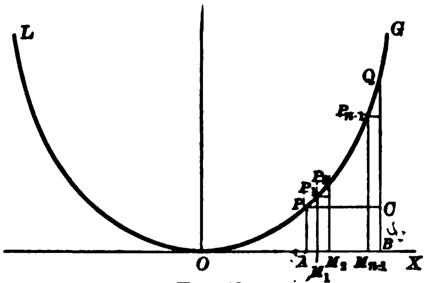

FIG 40.

$$n\,\Delta x = b - a;$$

draw ordinates at the points of division, and construct the set of inner rectangles and the set of outer rectangles. As in (*a*), it can be seen that sum of inner rectangles $<$ area $APQB <$ sum of outer rectangles; and also that

$$(\text{sum of outer rectangles}) - (\text{sum of inner rectangles}) = CQ \cdot \Delta x,$$

which approaches zero when Δx approaches zero. Hence the area $APQB$ is the limit of the sum of either set of rectangles when Δx approaches zero. The limit of the sum of the inner rectangles will now be found.

At A, $\quad x = a,\quad$ and hence, $\quad AP = a^2$;

at M_1, $\quad x = a + \Delta x,\quad$ and hence, $\quad M_1P_1 = (a + \Delta x)^2$;

at M_2, $\quad x = a + 2\,\Delta x,\quad$ and hence, $\quad M_2P_2 = (a + 2\,\Delta x)^2$;

- -

at M_{n-1}, $\quad x = a + \overline{n-1}\cdot\Delta x$, and hence, $M_{n-1}P_{n-1} = (a + \overline{n-1}\cdot\Delta x)^2$.

$$\therefore \text{ sum of inner rectangles} = a^2\Delta x + (a+\Delta x)^2\Delta x + (a + 2\,\Delta x)^2\Delta x + \cdots + (a + \overline{n-1}\cdot\Delta x)^2\Delta x.$$

$$\therefore \text{ area } APQB = \lim_{\Delta x \doteq 0}\{a^2 + (a+\Delta x)^2 + (a+2\,\Delta x)^2 + \cdots + (a + \overline{n-1}\cdot\Delta x)^2\}\Delta x$$

$$= \lim_{\Delta x \doteq 0}\{na^2 + 2\,a\,\Delta x(1 + 2 + 3 + \cdots + \overline{n-1}) + (\Delta x)^2(1^2 + 2^2 + 3^2 + \cdots + \overline{n-1}^2)\}\Delta x.$$

Now $\qquad 1 + 2 + 3 + \cdots + \overline{n-1} = \frac{1}{2}\,n(n-1)$;

and $\qquad 1^2 + 2^2 + 3^2 + \cdots + \overline{n-1}^2 = \frac{1}{6}\,(n-1)n(2\,n-1)$.*

$$\therefore \text{ area } APQB = \lim_{\Delta x \doteq 0} n\,\Delta x\{a^2 + an\,\Delta x - a\,\Delta x + \tfrac{1}{3}\,(n\,\Delta x)^2 - \tfrac{1}{2}\,n\,(\Delta x)^2 + \tfrac{1}{6}\,(\Delta x)^2\}.$$

* It is shown in algebra that the sum of the squares of the first n natural numbers, viz. 1^2, 2^2, 3^2, $\cdots$, n^2, is $\frac{1}{6}\,n(n+1)(2\,n+1)$.

But $$n\,\Delta x = b - a, \text{ no matter what } n \text{ and } \Delta x \text{ may be.}$$

$$\begin{aligned}\therefore \text{ area } APQB &= \lim_{\Delta x \doteq 0}(b-a)\{a^2 + a(b-a) - a\,\Delta x + \tfrac{1}{3}(b-a)^2 \\ &\qquad + \tfrac{1}{2}(b-a)\Delta x + \tfrac{1}{6}(\Delta x)^2\} \\ &= (b-a)\left(\frac{b^2+ab+a^2}{3}\right) \\ &= \frac{b^3}{3} - \frac{a^3}{3}.\end{aligned}$$

NOTE 1. In this example *the element of area* is a rectangle of height y and width Δx, when Δx becomes infinitesimal, *i.e.* the element of area is $y\,dx$, *i.e.* $x^2\,dx$, in which $dx \doteq 0$.

NOTE 2. It may be observed in passing that the result (1) can be obtained by taking the anti-differential of $x^2\,dx$, namely $\frac{x^3}{3}$, substituting b and a in turn for x therein, and calculating the difference $\frac{b^3}{3} - \frac{a^3}{3}$.

EX. Find the limit of the sum of outer rectangles.

(c) *Find the distance through which a body falls from rest in t_1 seconds, it being known that the speed acquired in falling for t seconds is gt feet per second.* [*Here g represents a number whose approximate value is* 32.2.]

NOTE 1. If the speed of a body is v feet per second and the speed remains uniform, the distance passed over in t seconds is vt feet.

Let the time t_1 seconds be divided into n intervals each equal to Δt, so that

$$n\,\Delta t = t_1.$$

The speed of the falling body *at the beginning* of each of these successive intervals of time is

$$0,\ g\cdot\Delta t,\ 2g\cdot\Delta t,\ \cdots,\ (n-1)g\cdot\Delta t, \text{ respectively};$$

the speed of the falling body *at the end* of each successive interval of time is

$$g\cdot\Delta t,\ 2g\cdot\Delta t,\ 3g\cdot\Delta t,\ \cdots,\ ng\cdot\Delta t, \text{ respectively.}$$

For any interval of time the speed of the falling body at the beginning is less, and the speed at the end is greater, than the speed at any other moment of the interval. Now let the distance be computed which would be passed over by the body if it successively had the speeds at the beginnings of the intervals; and then let the distance be computed which would be passed over by the body if it successively had the speeds at the ends of the intervals.

$$\begin{aligned}\text{The first distance } &= 0 + g(\Delta t)^2 + 2g(\Delta t)^2 + \cdots + (n-1)g(\Delta t)^2 \\ &= [0+1+2+\cdots+(n-1)]g(\Delta t)^2 \\ &= \tfrac{1}{2}n(n-1)g(\Delta t)^2.\end{aligned}$$

The second distance $= [1 + 2 + 3 + \cdots + n]g(\Delta t)^2$

$$= \tfrac{1}{2} n(n + 1)g(\Delta t)^2.$$

The actual distance fallen through, which may be denoted by s, evidently lies between these two distances; *i.e.*

$$\tfrac{1}{2} n(n - 1)g(\Delta t)^2 < s < \tfrac{1}{2} n(n + 1)g(\Delta t)^2.$$

On putting t_1 for its equal, $n\,\Delta t$, this becomes

$$\tfrac{1}{2} gt_1^2 - \tfrac{1}{2} gt_1 \cdot \Delta t < s < \tfrac{1}{2} gt_1^2 + \tfrac{1}{2} gt_1 \cdot \Delta t.$$

On letting Δt approach zero these three distances approach equality, and hence

$$s = \tfrac{1}{2} gt_1^2.$$

NOTE 2. For two other examples see Art. 96, Note 4.

96. Integration as summation. The definite integral. It will now be shown, **geometrically**, how integration is a process of summation. Let $f(x)$ denote any function of x which is continuous from $x = a$ to $x = b$ and geometrically representable. Let its graph be the curve K whose equation is accordingly

$$y = f(x).$$

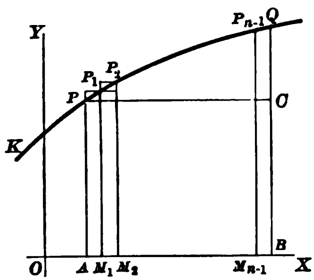

FIG. 41.

Suppose that $OA = a$ and $OB = b$, and draw the ordinates AP and BQ. Divide AB into n parts, each equal to Δx; accordingly,

$$n\,\Delta x = b - a. \qquad (1)$$

At the points of division erect ordinates, and construct inner and outer rectangles as in Art. 95 (a), (b). It can be shown, as in the examples in Art. 95, that the difference between the set of the inner rectangles and the set of the outer rectangles is $CQ \cdot \Delta x$ (CQ being equal to $BQ - AP$), a difference which approaches zero when Δx approaches zero. The area $APQB$ lies between these sets and evidently is the limit of the sum of either set of rectangles when Δx approaches zero. The limit of the sum of inner rectangles will now be found.

At A, $x = a$, and hence, $AP = f(a)$;

at M_1, $x = a + \Delta x$, and hence, $M_1P_1 = f(a + \Delta x)$;

at M_2, $x = a + 2\,\Delta x$, and hence, $M_2P_2 = f(a + 2\,\Delta x)$;

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at M_{n-1}, $x = b - \Delta x$, and hence, $M_{n-1}P_{n-1} = f(b - \Delta x)$.

$\therefore$ area $APQB = \lim_{\Delta x \doteq 0}$

$$\{f(a)\Delta x + f(a + \Delta x)\Delta x + f(a + 2\,\Delta x)\Delta x + \cdots + f(b - \Delta x)\Delta x\}. \quad (2)$$

The second member, which is the sum of the values, infinite in number, that $f(x)\Delta x$ takes when x increases from a to b by equal infinitesimal increments Δx, may be written (*i.e.* denoted by)

$$\lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} f(x)\,\Delta x.\text{*}$$

It is the custom, however, to denote the second member of (2) by putting the old-fashioned long S before $f(x)dx$ and writing at the bottom and top of the S respectively the values of x at which the summation begins and ends; thus

$$\int_{x=a}^{x=b} f(x)dx; \text{ or, more briefly, } \int_a^b f(x)dx. \quad (3)$$

This symbol is read "the integral of $f(x)\,dx$ between the limits a and b," or "the integral of $f(x)dx$ from $x = a$ to $x = b$."

NOTE 1. The numbers a and b are usually called the *lower* and *upper limits of* x. It would be better, perhaps, not to use the word *limit* in this connection, but to say "the initial and final values of x," or simply, "the end-values of x." †

NOTE 2. The infinitesimal differential $f(x)dx$ is called *an element* of the integral. It is the area of an infinitesimal rectangle of altitude $f(x)$ and infinitesimal base dx.

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* The latter part of this symbol denotes, and is to be read, "the sum of all quantities of the type" [or "form"] "$f(x)\Delta x$, from $x = a$ to $x = b$" [or "between $x = a$ and $x = b$"].

† Joseph Fourier (1768–1830) first devised the way shown in (3) of indicating the end-values of x.

NOTE 3. It is not necessary that the infinitesimal bases, *i.e.* the increments Δx of x, be all equal; but for purposes of elementary explanation it is somewhat simpler to take them as all equal. (See Lamb, *Calculus*, Arts. 86, 87, and the references in Art. 97, Note 5; also Snyder and Hutchinson, *Calculus*, Art. 150.)

NOTE 4. For the calculation of $\int_a^x e^x dx$ and $\int_a^x \sin x\, dx$ by the process shown in Art. 95, see Echols, *Calculus*, Art. 125.

The sum in brackets in (2) will now be calculated, and then its limit, which is indicated by the symbol (3), will be found.

Let the anti-differential (Art. 27 *a*) of $f(x)dx$* be denoted by $\phi(x)$; that is, let

$$f(x)\, dx = d\phi(x).$$

Then, by the elementary principle of differentiation (see Art. 22, Note 3) for all values of x from a to b,

$$\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x} = f(x) + e, \qquad (4)$$

in which e denotes a function whose value varies with the value of x, and which approaches zero when Δx approaches zero. On clearing of fractions and transposing, (4) becomes

$$f(x)\,\Delta x = \phi(x+\Delta x) - \phi(x) - e\cdot\Delta x. \qquad (5)$$

On substituting $a, a+\Delta x, a+2\,\Delta x, \cdots, b-\Delta x$ in turn for x in (5), and denoting the corresponding values of e by $e_1, e_2, e_3, \cdots, e_n$, respectively, there is obtained:

$$\begin{aligned}
f(a)\,\Delta x &= \phi(a+\Delta x) - \phi(a) - e_1\cdot\Delta x,\\
f(a+\Delta x)\,\Delta x &= \phi(a+2\,\Delta x) - \phi(a+\Delta x) - e_2\cdot\Delta x,\\
f(a+2\,\Delta x)\,\Delta x &= \phi(a+3\,\Delta x) - \phi(a+2\,\Delta x) - e_3\cdot\Delta x,\\
&\ \ \vdots\\
f(b-\Delta x)\,\Delta x &= \phi(b) - \phi(b-\Delta x) - e_n\cdot\Delta x.
\end{aligned}$$

* If $f(x)$ is a continuous function of x, $f(x)\,dx$ has an anti-differential. For proof see Picard, *Traité d'Analyse*, t. I. No. 4; also see Echols, *Calculus*, Appendix, Note 9.

Addition gives

$$f(a)\,\Delta x + f(a+\Delta x)\,\Delta x + f(a+2\,\Delta x)\,\Delta x + \cdots + f(b-\Delta x)$$
$$= \phi(b) - \phi(a) - (e_1 + e_2 + e_3 + \cdots + e_n)\,\Delta x. \qquad (6)$$

On taking the limit of each member of (6) when Δx approaches zero,

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) - \lim_{\Delta x \doteq 0}(e_1 + e_2 + \cdots + e_n)\,\Delta x. \qquad (7)$$

Let e_1 be one of the e's which has an absolute value E not less than any of the others; then evidently

$$(e_1 + e_2 + \cdots + e_n)\,\Delta x < nE\Delta x;$$

i.e. by (1), $\qquad (e_1 + e_2 + \cdots + e_n)\,\Delta x < (b-a)\,E.$

Hence, $\lim_{\Delta x \doteq 0}(e_1 + e_2 + \cdots + e_n)\,\Delta x = 0$, since E approaches zero when Δx approaches zero; and therefore,

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a). \qquad (8)$$

That is, expressing (8) in words: *The integral* $\int_a^b f(x)\,dx$, *which is the limit of the sum of all the values, infinite in number, that* $f(x)\,dx$ *takes as* x *varies by infinitesimal increments from* a *to* b, *is obtained by finding the anti-differential,* $\phi(x)$, *of* $f(x)\,dx$, *and then calculating* $\phi(b) - \phi(a)$.

NOTE 5. Many practical problems, such as finding areas, lengths of curves, volumes and surfaces of solids, and so on, can be reduced to finding the limit of the sum of an infinite number of infinitesimals of the form $f(x)\,dx$. (See Arts. 111, 112, 135–140.) As has been seen above, the anti-differential of $f(x)\,dx$ is of great service in determining this limit; accordingly, considerable attention must be given to mastering methods for finding anti-differentials.

NOTE 6. The process of finding the anti-differential of $f(x)\,dx$ is nearly always more difficult than the direct process of differentiation, and frequently the deduction of an anti-differential is impossible. When the anti-differential of $f(x)\,dx$ cannot be found in a finite form in terms of ordinary functions, approximate values of the definite integral can be found by methods discussed in Chapter XIV. The impossibility of evaluating the first member of (8) in terms of the ordinary functions has sometimes furnished an occasion for defining a new function, whose properties are investigated in higher mathematics. (On this point see Snyder and Hutchinson, *Calculus*, Art. 123,

foot-note.) For instance, the subject of *Elliptic Functions* arose out of the study of what are called the *elliptic integrals* (see Art. 137, Ex. 4, Art. 174, Note 4, Art. 122, Note 4).

(The ordinary elementary functions can be defined by means of the calculus, and their properties thence developed.)

NOTE 7. At the beginning of this article the principle was enunciated that the area bounded by a smooth curve PQ (Fig. 41), the x-axis, and a pair of ordinates, is the limit of the sum of certain inner, or outer, rectangles constructed between the ordinates. The student can easily show that this principle holds for the smooth curves in Figs. 42 *a*, *b*, *c*.

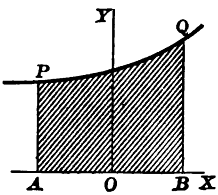

FIG. 42 *a*.

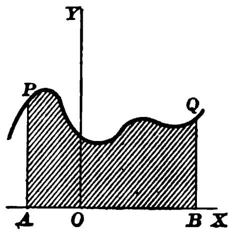

FIG. 42 *b*.

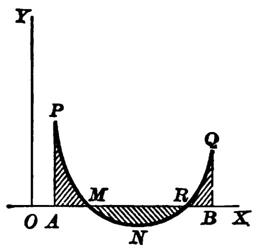

FIG. 42 *c*.

Note 8. This article shows that a definite integral may be represented geometrically as an area. For a general **analytical** *exposition of integration as a summation*, see Snyder and Hutchinson, *Calculus*, Art. 148. Their exposition depends on Taylor's theorem (Art. 176). Also see the references mentioned in Art. 97, Note 5.

Ex. Show that the calculus method of computing the area in Fig. 42 *c* bounded by $PMNRQ$, AB, AP, and BQ really gives area APM + area RQB − area MNR.

[As a point moves along the curve from P to Q, dx is always positive. In APM y is positive, in MNR negative, in RQB positive. Accordingly, the elements of area, $f(x)\,dx$ or $y\,dx$, are positive in APM and RQB, and negative in MNR.]

EXAMPLES.

N.B. The knowledge already obtained in Chapter IV. about anti-differentials is sufficient for the solution of the following examples. *It is advisable to make drawings* of the curves and the figures whose areas are required.

1. Find the area between the cubical parabola $y = x^3$ (Fig., p. 412), the x-axis, and the ordinates for which $x = 1$, $x = 3$.

According to (3) and (8), the area required $= \int_1^3 x^3\,dx$

$$= \left[\frac{x^4}{4} + c\right]_1^3$$

$$= \tfrac{81}{4} + c - (\tfrac{1}{4} + c)$$

$$= 20 \text{ sq. units of area.}$$

2. Find the area between the curve in Ex. 1, the x-axis, and the ordinates for which $x = -2$, $x = 3$. *Ans.* $16\frac{1}{4}$ sq. units.

3. Explain the apparent contradiction between the results in Exs. 1, 2.

4. Find the actual number of square units in the figure whose boundaries are given in Ex. 2. *Ans.* $24\frac{1}{4}$ sq. units.

5. Find the area between the parabola $2y = 7x^2$, the x-axis, and the ordinates for which: (1) $x = 2$, $x = 4$; (2) $x = -3$, $x = 5$.

Ans. (1) $65\frac{1}{3}$ sq. units; (2) $177\frac{1}{3}$ sq. units.

N.B. A table of square roots will save time and trouble.

6. Find the area between the parabola $y^2 = 8x$, the x-axis, and the ordinates for which: (1) $x = 0$, $x = 3$; (2) $x = 2$, $x = 7$.

Ans. (1) 9.798 sq. units; (2) 29.59 sq. units.

7. Find the area of the figure bounded by the parabola $y^2 = 6x$ and the chord perpendicular to the x-axis at $x = 4$. *Ans.* 26.128 sq. units.

8. Find, by the calculus, the area bounded by the line $y = 3x$, the x-axis, and the ordinate for which $x = 4$. *Ans.* 24 sq. units.

9. (1) Find, by the calculus, the area of the figure bounded by the line $y = 3x$, the x-axis, and the ordinates for which $x = 4$, $x = -4$. (2) How many sq. units of gold leaf are required to cover this figure?

Ans. (1) 0; (2) 48 sq. units.

10. (1) Find the area between a semi-undulation of the curve $y = \sin x$ and the x-axis. (2) Find the area of the figure bounded by a complete undulation of this curve and the x-axis. (3) How many sq. units of gold-leaf are required to cover this figure. *Ans.* (1) 2; (2) 0; (3) 4.

11. Compute the area enclosed by the parabola $y^2 = 4x$ and the lines $x = 2$, $x = 5$. *Ans.* 22.27 sq. units.

12. Compute the area enclosed by the parabola $y = x^2$ and the lines $y = 1$, $y = 4$. *Ans.* $9\frac{1}{3}$ sq. units.

13. Find the area between the parabolas $x^2 = y$ and $y^2 = 8x$.

Ans. $2\frac{2}{3}$ sq. units.

14. Find the area between the curves: (1) $y^2 = x$ and $y^2 = x^3$; (2) $x^2 = y$ and $y^2 = x^3$. (Make figures.) *Ans.* (1) $\frac{4}{15}$ sq. units; (2) $\frac{1}{15}$ sq. units.

15. Find the area bounded by the curves in Ex. 14 (2) and the lines $x = 2$, $x = 4$. *Ans.* 8.129 sq. units.

N.B. Art. 111 may be taken up now.

97. Integration as the inverse of differentiation. The indefinite integral. Constant of integration. Particular integrals. In many cases there is required, not the limit of the sum of an infinite number of infinitesimals of the form $f(x)dx$, but the function whose derivative or differential is given. The following is an instance from geometry. When a curve's equation, $y=f(x)$, is known, differentiation gives the slope at any point on the curve in terms of the abscissa x, namely, $\frac{dy}{dx}=f'(x)$ (Art. 24). On the other hand, if this slope is given, integration affords a means of finding the equation of the curve (or curves) satisfying the given condition as to slope. Again, an instance from mechanics: if a quantity changes with time in an assigned way, differentiation determines the rate of change for any instant (Art. 25). On the other hand, if this rate of change is known, integration provides a means for determining the quantity in terms of the time. (See Art. 22, Notes 1, 2, and Art. 27 *a*.)

EXAMPLES.

Ex. 1. The slope at any point (x, y) of the cubical parabola $y = x^3$ is $3x^2$; that is, at all points on this curve, $\frac{dy}{dx} = 3x^2$ and $dy = 3x^2\,dx$.

Now suppose it is known that a curve satisfies *the following condition*, namely, that its slope at any point (x, y) is $3x^2$; *i.e.* that for this curve,

$$\frac{dy}{dx} = 3x^2, \text{ (whence, } dy = 3x^2\,dx).$$

Then, evidently, $\qquad y = x^3 + c,$

in which c is a constant which can take any arbitrarily assigned value. This number c is called *a constant of integration;* its geometrical meaning is explained in Art. 99. Since c denotes *any* constant, there is evidently an infinite number of curves (cubical parabolas, $y = x^3 + 2$, $y = x^3 - 10$, $y = x^3 + 7$, etc., etc.) which satisfy the given condition. If *a second condition* is imposed, the constant c will have a definite and particular value. For instance, let the curve be required to pass through the point $(2, 1)$. Then, $1 = 2^3 + c$; whence $c = -7$, and the equation of the curve satisfying both the conditions above is $y = x^3 - 7$. (Also see Ex. 17, Art. 37.)

2. Suppose that a body is moving in a straight line in such a way that (the number of units in) its distance from a fixed point on the line is always

(the number of units in) the logarithm of the number of seconds, t say, since the motion began; *i.e.* so that

$$s = \log t.$$

Then, the speed, $\frac{ds}{dt} = \frac{1}{t}$, and $ds = \frac{dt}{t}$.

Now suppose it is known that at any time after the beginning of its motion, after t seconds say, the speed of a moving body is $\frac{1}{t}$; *i.e.* that

$$\frac{ds}{dt} = \frac{1}{t}, \left(\text{whence, } ds = \frac{dt}{t}\right).$$

Then, evidently, $s = \log t + c$,

in which c is an arbitrary constant. If a second condition is imposed, the constant c will take a definite value. For instance, let the body be 4 units from the starting-point at the end of 2 seconds, *i.e.* let $s = 4$ when $t = 2$. Then

$$4 = \log 2 + c; \text{ whence } c = 4 - \log 2,$$

and $s = \log t + 4 - \log 2.$

3. In Ex. 1 determine c so that the cubical parabola shall go through (a) the point (0, 0); (b) the point (7, -4); (c) the point (-8, 2); (d) the point (h, k). Draw the curves for (a), (b), (c).

4. Find the curves for which the slope at any point is 4. Determine the particular curves which pass through the points (0, 0), (2, 3), (-7, 1), respectively. Draw these curves.

5. Find the curves for which (the number of units in) the slope at any point is 8 times (the number of units in) the abscissa of the point. Determine the particular curves which pass through the points (0, 0), (1, 2), (2, 3), (-4, 2), respectively. Draw these curves.

6. How are the curves in Exs. 4, 1, 3, 5, respectively, affected when the constants of integration are changed?

7. If at any moment the velocity in feet per second at which a body is falling is 32 times the number of seconds elapsed since it began to fall from rest, what is *the general formula* for its distance, at any instant, from a point on the line of fall?

In this instance, $\frac{ds}{dt} = 32\,t$, (whence, $ds = 32\,t\,dt$).

Hence $s = 16\,t^2 + c.$

8. In Ex. 7, at the end of t seconds what is the distance measured from the starting-point? What is the distance at the end of 2 seconds? of 4 seconds? of 5 seconds? What are the distances, in these respective distances, measured from a point 10 feet above the starting-point? If at the time of the beginning of fall, the body is 20 feet below the point from which

distance is measured, what is its distance below this point at the end of t seconds? Explain the meaning of the constant of integration in the general formula derived in Ex. 7? Derive the results in Ex. 8 from this general formula.

Suppose that $$d\phi(x) = f(x)dx; \tag{1}$$

then also (Art. 29), $$d\{\phi(x) + c\} = f(x)dx, \tag{2}$$

in which c is any constant. Hence, if $\phi(x)$ is an anti-differential of $f(x)dx$, $\phi(x) + c$ is also an anti-differential of $f(x)dx$. That is,

if $$d\phi(x) = f(x)dx,$$

then $$\int f(x)dx = \phi(x) + c, \tag{3}$$

in which c is an arbitrary constant. Thus the anti-differential of $f(x)dx$ is *indefinite*, so far as an added arbitrary constant is concerned. (This has already been pointed out in Art. 29, Note 6.) On this account the anti-differential is called **the indefinite integral.** The arbitrary constant is called **the constant of integration.** The indefinite integral is often called *the general integral.* If the constant of integration be given a particular value, as $\frac{1}{2}$, -2, 100, etc., the integral is called **a particular integral.** For instance, the indefinite, or general, integral of x^5dx, *i.e.* $\int x^5dx$ is $\frac{1}{6}x^6 + c$; and particular integrals of x^5dx are $\frac{1}{6}x^6 + 5$, $\frac{1}{6}x^6 - 11$, etc.

9. Name the indefinite (or general) integrals and the particular integrals appearing in Exs. 1–8.

10. How many particular integrals (anti-differentials) can a function have? What must the difference between any pair of them be?

NOTE 1. It should be noted that the indefiniteness in the integral does not extend to the terms involving the variable. For instance,

$$\int (x+1)dx = \tfrac{1}{2}x^2 + x + c,$$

and $\int (x+1)dx = \int (x+1)d(x+1)^* = \frac{1}{2}(x+1)^2 + k = \frac{1}{2}x^2 + x + 1 + k$; thus the terms involving x are the same.

NOTE 2. The origin of the words *integral* and *integration* has been indicated in Art. 94. It is, in a measure, to be regretted that the term *integral* and the symbol $\int$, which both imply summation, should also be used to denote an anti-differential. In accordance with the fashion in vogue

* Since $d(x+1) = dx$.

in trigonometry for denoting inverse functions (*e.g.* $\sin x$ and $\sin^{-1} x$ for sine of x and anti-sine, or inverse sine, of x, respectively *) the anti-derivative of $f(x)$ and the anti-differential of $f(x)dx$ are sometimes denoted by $D^{-1}f(x)$ and $d^{-1}f(x)dx$ respectively. Thus $\int f(x)dx$, $d^{-1}f(x)dx$, and $D^{-1}f(x)$, are equivalent.

NOTE 3. If $d\phi(x) = f(x)dx$, then (Art. 96) $\int_a^x f(x)dx = \phi(x) - \phi(a)$. If the upper end-value x is variable, and the lower end-value a is arbitrary, then this integral is indefinite and of the form $\phi(x) + c$. Accordingly, the indefinite integral may be regarded as in the form of a definite integral whose upper end-value is the variable, and whose lower end-value is arbitrary.

NOTE 4. Result (8), Art. 96 for the area of $APQB$ (Fig. 41) can also be derived by a method which is founded on the notion of the indefinite integral. For instance, see Todhunter, *Integral Calculus*, Art. 128, or Murray, *Integral Calculus*, Art. 13.

NOTE 5. **References for collateral reading** *on the notions of integration, definite integral, and indefinite integral.* Gibson, *Calculus*, §§ 82, 110, 124–126 ; Williamson, *Integral Calculus*, Arts. 1, 90, 91, 126 ; Harnack, *Calculus* (Cathcart's translation), §§ 100–106 ; Echols, *Calculus*, Chap. XVI.; Lamb, *Calculus*, Arts. 71, 72, 86–93.

98. Geometric or graphical representation of definite integrals. Properties of definite integrals. It has been seen (Art. 96) that if PQ (Fig. 41) is the curve whose equation is

$$y = f(x),$$

then the integral $$\int_a^b f(x)\,dx$$

gives the area bounded by the curve, the x-axis, and the ordinates for which $x = a$ and $x = b$ respectively. Accordingly, the figure thus bounded may be said, and may be used, to represent the integral graphically. Hence, in order to represent an integral, $\int_l^m \phi(x)\,dx$ say (no matter whether this integral be an area, or a length, or a volume, or a mass, etc.), draw the curve whose equation is $y = \phi(x)$, and draw the ordinates for which $x = l$ and $x = m$ respectively. The figure bounded by the curve, the x-axis, and these ordinates, is the graphical representative of the integral, and (Art. 96) the number of units in the area of this figure is the same as the number of units in the integral.

* See Art. 12, Note.

The following **properties of definite integrals** *are important.* Properties (b) and (c) are easily deduced by using the graphical representatives of the integrals.

(***a***) If $d\phi(x) = f(x)\,dx$, then (Art. 96)

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \quad \text{and} \quad \int_b^a f(x)\,dx = \phi(a) - \phi(b);$$

and hence, $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Therefore, if the end-values of the variable in an integral be interchanged, the algebraic sign of the integral will be changed.

Ex. Give several concrete illustrations of this property.

(***b***) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ whatever c may be.

Draw the curve $y = f(x)$, and draw ordinates AP, BQ, CR, for which $x = a$, $x = b$, $x = c$, respectively. Then:

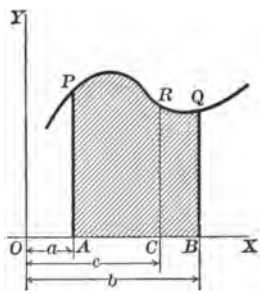

FIG. 43 *a*.

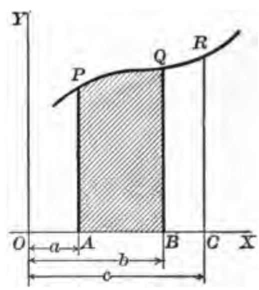

FIG. 43 *b*.

In Fig. 43 a,

$$\int_a^b f(x)\,dx = \text{area } APQB$$
$$= \text{area } APRC + \text{area } CRQB$$
$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In Fig. 43 b,

$$\int_a^b f(x)\,dx = \text{area } APQB$$
$$= \text{area } APRC - \text{area } BQRC$$
$$= \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$
$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Similarly, it can be shown that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \cdots + \int_k^l f(x)\,dx + \int_l^b f(x)\,dx.$$

That is, a definite integral can be broken up into any number of similar definite integrals that differ only in their end-values. (Similar definite integrals are those in which the same integrand appears.)

Ex. 1. Prove the principle just enunciated.

Ex. 2. Give concrete illustrations of the principles in (b).

(c) *The mean value of $f(x)$ for all values of x from a to b.* (That is, the mean value of $f(x)$ when x varies continuously

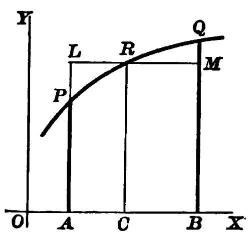

FIG. 44 a.

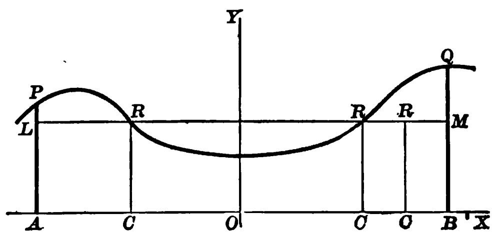

FIG. 44 b.

from a to b.) Draw the curve $y = f(x)$, and at A and B erect the ordinates for which $x = a$ and $x = b$ respectively. Then

$$\int_a^b f(x)\,dx = \text{area } APQB.$$

Now, evidently, on the base AB there can be a rectangle whose area is the same as the area of $APQB$. Let $ALMB$, which has an altitude CR, be this rectangle; then

$$\int_a^b f(x)\,dx = \text{area } ALMB = \text{area } AB \cdot CR$$

$$= (b - a) \cdot \text{length } CR. \qquad (1)$$

The length CR is said to be the mean value of the ordinates $f(x)$ from $x = a$ to $x = b$. Hence, from (1),

$$\left.\begin{matrix}\textbf{Mean value of } f(x) \textbf{ from} \\ x = a \textbf{ to } x = b\end{matrix}\right\} = \frac{\int_a^b f(x)\,dx}{b-a}.\text{*} \qquad (2)$$

In words, *the mean value of $f(x)$ when x varies continuously from a to b, is equal to the integral of $f(x)dx$ from the end-value a to the end-value b, divided by the difference between these end-values.*

EXAMPLES.

1. Make a graphical representation of each of the integrals appearing in Exs. 2–5 below.

2. Find the mean length of the ordinates of the parabola $y = x^2$ from $x = 1$ to $x = 3$.

$$\text{Mean length} = \frac{\int_1^3 x^2\,dx}{3-1} = 4\tfrac{1}{3}.$$

3. In the parabola $y = x^2$, find the mean length of the ordinates of the arc between $x = 0$ and $x = 2$; and find the mean length of the ordinates from $x = -2$ to $x = 2$. Explain, with the help of a figure, why these mean lengths are the same.

4. In the cubical parabola $y = x^3$.

5. In the line $y = 4x$.

99. Geometric (or graphical) representation of indefinite integrals. Geometric meaning of the constant of integration. If

$$d\,\phi(x) = f(x)\,dx,$$

then (Art. 97)
$$\int f(x)\,dx = \phi(x) + c, \qquad (1)$$

in which c is an arbitrary constant. Draw the curve

$$y = \phi(x); \qquad (2)$$

let AB be the curve. Give c the particular values 2 and 10, and draw the curves,
$$y = \phi(x) + 2 \qquad (3)$$
and
$$y = \phi(x) + 10. \qquad (4)$$

---

* For clear proof that this is the mean value, see Art. 141, where the topic of mean values is more fully discussed, and Echols, *Calculus*, Art. 150 (and Arts. 151, 152).

Let CD and EF be these curves. In the case of each one of the curves obtained by giving particular values to c,

$$\frac{dy}{dx} = f(x);$$

and hence, at points having the same abscissa the tangents to these curves have the same slope, and, accordingly, are parallel. For instance, on each curve, at the point whose abscissa is m the slope of the tangent is $f(m)$.

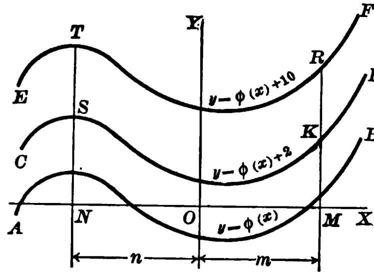

FIG. 45.

Moreover, the distance between any two curves obtained by giving c particular values, measured along any ordinate, is always the same. For, draw the ordinates KR and ST at $x = m$ and $x = n$, respectively, as in the figure. Then, by Equations (3) and (4),

$$MK = \phi(m) + 2; \quad \text{and} \quad NS = \phi(n) + 2;$$
$$MR = \phi(m) + 10; \quad NT = \phi(n) + 10.$$

Hence $\quad KR = 8, \quad$ and $\quad ST = 8.$

Accordingly, the graphical representation of the indefinite integral, $\int f(x)\,dx$, consists of the family of curves, infinite in number, whose equations are of the form $y = \phi(x) + c$, and which are severally obtained by giving c particular values; and the effect of changing c is to move the curve in a direction parallel to the y-axis. (Also see Art. 29, Note 2.)

Ex. 1. How many different values can be assigned to c? How many particular integrals are included in the general integral? How many different curves can represent the indefinite integral?

Ex. 2. Write the equations of several curves representing each of the following integrals, viz.: $\int x\,dx$, $\int x^2\,dx$, $\int 3x\,dx$, $\int 3\,dx$, $\int (2x+5)\,dx$. Draw the curves.

100. Integral curves. If $d\,\phi(x) = f(x)\,dx$,

then (Art. 96) $$\int_0^x f(x)\,dx = \phi(x) - \phi(0).$$

The curve whose equation is

$$y = \phi(x) - \phi(0), \text{ i.e. } y = \int_0^x f(x)\,dx, \tag{1}$$

which is one of the particular curves representing $y = \phi(x) + c$ (see Art. 99), is called *the first integral curve* for the curve $y = f(x)$. Since the area of the figure bounded by the curve $y = f(x)$, the x-axis, and the ordinates at $x = 0$ and $x = x$, is $\phi(x) - \phi(0)$ (Art. 96), the number of units of length in the ordinate at the point of abscissa x on the curve (1), is the same as the number of units of area in this figure. Accordingly, if the first integral curve of a given curve be drawn, the area bounded by the given curve, the axes, and the ordinate at any point on the x-axis, can be obtained merely by measuring the length of the ordinate drawn from the same point to the integral curve. Consequently, it may be said that this ordinate graphically represents the area, and thus, the integral.

Note 1. The original curve $y = f(x)$ is the derived or differential curve of curve (1).

Ex. For instance, for the line $y = \frac{1}{2}x + 3$; (2)

since $$\int_0^x (\tfrac{1}{2}x + 3)\,dx = \tfrac{1}{4}x^2 + 3x,$$

the first integral curve of curve (2) is the parabola $y = \frac{1}{4}x^2 + 3x$. (3)

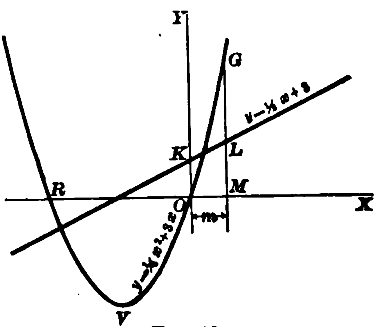

Fig. 46.

These two curves are shown here. If M be any point on the x-axis, and $OM = m$ units of length, and the ordinate MLG be drawn,

(the number of units of length in MG) = (the number of units of area in $OKLM$).

For, length MG, by (3), is $\frac{1}{4}m^2 + 3m$; and

area $OKLM$

$$= \int_0^m (\tfrac{1}{2}x + 3)\,dx = \tfrac{1}{4}m^2 + 3m.$$

Just as a given curve — it may be called the original or *the fundamental curve* — has a first integral curve, this first integral curve also has an integral curve. The latter curve is called *the second integral curve* of the fundamental curve. Again, the second integral curve has an integral curve; this is said to be *the third integral curve* of the fundamental curve. On proceeding in this way a system of any number of successive integral curves may be constructed belonging to a given fundamental curve.

NOTE 2. The integral curve can be drawn mechanically from its fundamental by means of an instrument called **the integraph,** invented by a Russian engineer, Abdank-Abakanowicz.

Note 3. Integral curves are of great assistance in obtaining graphical solutions of practical problems in mechanics and physics. For further information about integral curves and their uses and the theory of the integraph, and for other references, see Gibson, *Calculus*, §§ 83, 84; Murray, *Integral Calculus*, Art. 15, Chap. XII., pp. 190–200 (integral curves), Appendix, Note *G* (on integral curves), pp. 240–245; M. Abdank-Abakanowicz, *Les Intégraphes: la courbe intégrale et ses applications* (Paris, Gauthier-Villars), or *Bitterli's* German translation of the same, with additional notes (Leipzig, Teubner). Also see catalogues of dealers in mathematical and drawing instruments.

EXAMPLES.

1. Show that, for the same abscissa, the number of units of length in the ordinate of the fundamental curve is the same as the number of units in the slope of its first integral curve.

2. Does the first integral curve belong to the family of curves referred to in Art. 99?

3. Show how the members of the family of curves in Art. 99 may be easily drawn when an integraph is available.

4. Write the equations of the first, second, and third integral curves of the following curves: (*a*) $y = x$; (*b*) $y = 2x + 5$; (*c*) $y = \sin x$; (*d*) $y = e^x$. Draw all these fundamental and integral curves. Can the curve $x^3y = 1$ be treated in a similar manner?

5. Find and draw the curve of slopes for each of the curves (*a*), (*b*), (*c*), (*d*), Ex. 4. Then find and draw the first, second, and third integral curves of each of these curves of slope.

101. Summary. The two processes of the infinitesimal calculus, namely, differentiation and integration, have now been briefly described.

The process of **differentiation** is used in solving this problem, among others: the function of a variable being given, *find the limiting value of the ratio* of the increment of the function to the increment of the variable when the increment of the variable approaches zero (Art. 22). This problem is equivalent to *finding the ratio* of the rate of increase of the function to the rate of increase of the variable (Art. 26). If the function be represented by a curve, the problem is equivalent to *finding the slope of the curve at any point* (Art. 24).

The process of **integration** may be regarded as either:

(*a*) a process of summation; or

(*b*) a process which is the inverse of differentiation.

Integration is used in solving both of the following problems, viz.:

(1) *To find the limit of the sum* of infinitesimals of the form $f(x)\,dx$, x being given definite values at which the summation begins and ends (Arts. 94–96);

(2) *To find the anti-differential* of a given differential $f(x)\,dx$ (Art. 97).

Problem (1) is equivalent to finding a certain area; problem (2) is equivalent to finding a curve when its slope at every point is known.

In solving problem (1) the anti-differential of $f(x)\,dx$ is required (Art. 96). Hence, in both problems (1) and (2) it is necessary to find the anti-differentials of various functions of the form $f(x)\,dx$. Chapters XI. and XIII. are devoted to showing how anti-differentials may be found in the case of several of the comparatively small number of functions for which this is possible. It may be stated here that, in general, integration is more difficult than the direct process of differentiation.

CHAPTER XI.

ELEMENTARY INTEGRALS.

102. In this chapter the elementary or fundamental integrals (anti-differentials) are obtained, and some general theorems and particular methods which are useful in the process of anti-differentiation are described. There is one general fundamental process (Art. 22) by which the differential of a function can be obtained. On the other hand, there is no general process by which the anti-differential of a function can be found.* The simplest integrals, which are given in Art. 103, are discovered by means of results made known in differentiation.

In Art. 104 certain general theorems in integration are deduced. Two particular processes, or methods, of integration which are very serviceable and frequently used, are described in Arts. 105, 106. A further set of fundamental integrals is derived in Art. 107. When $f(x)$ is a rational fraction in x, the anti-differential of $f(x)dx$ may be found by means of the results in Arts. 103, 107; for this reason examples involving rational fractions are given in Art. 108. The integration of a total differential is considered in Art. 109.

So far as finding anti-differentials is concerned, this is the most important chapter in the book. The student is strongly recommended to make himself thoroughly familiar with the chapter and to work a large number of examples, so that he can apply its results readily and accurately. *The list of formulas, I. to XXVI. (Arts. 103, 107)*, **should be memorized.** Every function, $f(x)dx$, whose integral can be expressed in finite form in terms of the functions in elementary mathematics, is reducible to one or more of the forms in this list. It is often necessary to make reductions of this kind. A ready knowledge of these forms is not only useful

* There is a general process by which the value of a *definite* integral can be found approximately, as described in Art. 123.

for integrating them immediately when presented, but is also a great aid *in indicating the form at which to aim*, when it is necessary to reduce a complicated expression.

103. Elementary integrals. The following formulas in integration come directly from the results in Arts. 37–55, and can be verified by differentiation. Here u denotes a function of any variable, and c, c_0, c_1, denote arbitrary constants.

$$\text{I.} \quad \int u^n du = \frac{u^{n+1}}{n+1} + c, \text{ in which } n \text{ is a constant.}$$

NOTE 1. This result is applicable in the case of *all constant* values of n, excepting $n = -1$. The latter case is given in II.

$$\text{II.} \quad \int \frac{du}{u} = \log u + c_0 = \log u + \log c = \log cu.$$

NOTE 2. The various ways in which the constant of integration can appear in this integral, should be noted.

NOTE 3. Formula II. can also be derived by means of I. (See Murray, *Integral Calculus*, p. 37, foot-note.)

$$\text{III.} \quad \int a^u du = \frac{a^u}{\log a} + c.$$

$$\text{IV.} \quad \int e^u du = e^u + c.$$

$$\text{V.} \quad \int \sin u \, du = -\cos u + c.$$

$$\text{VI.} \quad \int \cos u \, du = \sin u + c.$$

$$\text{VII.} \quad \int \sec^2 u \, du = \tan u + c.$$

$$\text{VIII.} \quad \int \csc^2 u \, du = -\cot u + c.$$

$$\text{IX.} \quad \int \sec u \tan u \, du = \sec u + c.$$

$$\text{X.} \quad \int \csc u \cot u \, du = -\csc u + c.$$

$$\text{XI.} \quad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c = -\cos^{-1} u + c_1.$$

[REMARK. By trigonometry $\sin^{-1} u = -\cos^{-1} u + 2n\pi + \frac{\pi}{2}$. See Art. 97, Ex. 10 and Note 1.]

$$\text{XII.} \quad \int \frac{du}{1+u^2} = \tan^{-1} u + c.$$

$$\text{XIII.} \quad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + c.$$

$$\text{XIV.} \quad \int \frac{du}{\sqrt{2u-u^2}} = \text{vers}^{-1} u + c.$$

NOTE 4. Integrals XII., XIII., XIV., may also be written $-\cot^{-1} u + c$, $-\csc^{-1} u + c$, $-\text{covers}^{-1} u + c$, respectively.

104. General theorems in integration.

A. Let $f(x)$, $F(x)$, $\phi(x)$, $\cdots$, denote functions of x, finite in number. By Arts. 29, 31, 97, the differentials of

$$\int [f(x) + F(x) + \phi(x) + \cdots] dx + c_0 \text{ and}$$

$$\int f(x)dx + \int F(x)dx + \int \phi(x)dx + \cdots + c_1$$

are each $$f(x)\,dx + F(x)\,dx + \phi(x)dx + \cdots.$$

Hence, *the integral of the sum of a finite number of functions and the sum of the integrals of the several functions are the same in the terms depending on the variable, and can differ at most only by an arbitrary constant.*

(For integration of the sum of an *infinite* number of functions, see Art. 172.)

EXAMPLES.

1. $$\int (x^3 + \cos x + e^x)dx = \int x^3\,dx + \int \cos x\,dx + \int e^x\,dx + c_0$$
$$= \tfrac{1}{4}x^4 + \sin x + e^x + c. \quad (1)$$

NOTE 1. Each integral in the second member in Ex. 1 has an arbitrary constant of integration; but all these constants can be combined into one.

2. $\int (x^5 - \sin x + \sec^2 x)dx = \frac{1}{6}x^6 + \cos x + \tan x + c.$

B. The differentials of

$$\int mu\,dx + c_0 \text{ and } m\int u\,dx + c_1$$

are each $mudx$. Hence,

a constant factor can be moved from either side of the integration sign to the other without affecting the terms of the integral which depend on the variable.

C. The differentials of

$$\int u\,dx + c_0,\ m\int\frac{u}{m}\,dx + c_1,\ \frac{1}{m}\int mu\,dx + c_2,$$

are each $u dx$. Hence,

the terms of the integral which depend on the variable are not affected, if a constant is introduced at the same time as a multiplier on one side of the integration sign and as a divisor on the other.

NOTE 2. Theorems B and C are useful in simplifying integrations.

3. (1) $\int 3x\,dx = 3\int x\,dx = \frac{3}{2}x^2 + c.$

(2) $\int\frac{dx}{x^4} = \int x^{-4}\,dx = \frac{x^{-3}}{-4+1} + c = -\frac{1}{3x^3} + c.$

4. $\int 2\sin x\,dx = 2\int\sin x\,dx = -2\cos x + c.$

5. $\int\sin 2x\,dx = \frac{1}{2}\int 2\sin 2x\,dx = \frac{1}{2}\int\sin 2x\,d(2x) = -\frac{1}{2}\cos 2x + c.$

NOTE 3. *A factor involving the variable cannot be moved, or introduced, in the manner described in theorems B and C.* Thus, $\int x^2 dx = \frac{1}{3}x^3 + c$; but $x\int x\,dx = \frac{1}{2}x^3 + c$. Also, $\int x^2 dx = \frac{1}{3}x^3 + c$; but $\frac{1}{x}\int x^3 dx = \frac{1}{4}x^3 + c$.

6. $$\int\tan u\,du = \int\frac{\sin u}{\cos u}\,du = -\int\frac{d(\cos u)}{\cos u} = -\log(\cos u) + c$$
$$= \log(\sec u) + c.$$

7. $\int\cot u\,du = \int\frac{\cos u}{\sin u}\,du = \int\frac{d(\sin u)}{\sin u} + c = \log(\sin u) + c.$

8. $\int\frac{x^3\,dx}{7+x^4} = \frac{1}{4}\int\frac{d(7+x^4)}{7+x^4} = \frac{1}{4}\log(7+x^4) + c.$

9. Write the anti-derivatives of x^7, $6x^{72}$, $2x^{40}$, $4x^{-19}$, $5x^{-14}$, $\frac{1}{x^3}$, $\frac{12}{x^5}$, $3x^{\frac{4}{5}}$, $x^{\sqrt{2}}$, $6\sqrt[3]{x^5}$, $2\sqrt[5]{x^3}$, $\frac{4}{\sqrt{x}}$, $\frac{5}{\sqrt{x^3}}$, $\frac{6}{7\sqrt{x^{10}}}$.

10. Write the anti-differentials of $v^3\,dv$, $7\sqrt[3]{t^2}\,dt$, $\frac{2}{u^5}\,du$, $\frac{3}{\sqrt[4]{s^3}}\,ds$.

11. Find $\int ax^{\frac{m}{n}}\,dx$, $\int c\sqrt[3]{t^m}\,dt$, $\int l\sqrt[n]{v^6}\,dv$, $\int r\sqrt[s]{w^t}\,dw$.

12. $\int \frac{dv}{v}$, $\int \frac{2\,ds}{s+2}$, $\int \frac{x^5\,dx}{7-x^6}$, $\int \frac{(8t-3)}{4t^2-3t+11}\,dt$.

13. $\int e^t\,dt$, $\int 5\,e^{4x}\,dx$, $\int 4\,e^{x^2}\,x dx$, $\int 4^x\,dx$, $\int 10^{2x}\,dx$.

14. $\int \sin 3x\,dx$, $4\int \cos 7x\,dx$, $9\int \sec^2 5x\,dx$, $\int \sin(x+\alpha)\,dx$, $\int \cos(2x+\alpha)\,dx$, $\int \sec^2\left(\frac{3x}{5}+\frac{\pi}{2}\right)dx$.

15. $\int \sec 2x \tan 2x\,dx$, $\int \sec \frac{3}{4}x \tan \frac{3}{4}x\,dx$, $\int \frac{dt}{\sqrt{1-t^2}}$, $\int \frac{x\,dx}{\sqrt{1-x^4}}$, $\int \frac{7\,dx}{\sqrt{1-25x^2}}$, $\int \frac{5x^2\,dx}{\sqrt{1-x^6}}$, $\int \frac{dv}{\sqrt{1+v^2}}$, $\int \frac{t\,dt}{1+t^4}$, $\int \frac{2\,dx}{1+4x^2}$, $\int \frac{dt}{t\sqrt{t^2-1}}$, $\int \frac{dx}{x\sqrt{9x^2-1}}$, $\int \frac{x\,dx}{x^2\sqrt{x^4-1}}$, $\int \frac{dx}{\sqrt{6x-9x^2}}$, $\int \frac{dx}{\sqrt{8x-16x^2}}$.

16. $\int (t^2-4)^2\,dt$, $\int (a^{\frac{2}{3}}+x^{\frac{2}{3}})^3\,dx$, $\int e^{\frac{m}{n}x}\,dx$, $\int (\cos ax + \sin nx)\,dx$.

17. Express formula II. in words.

105. Integration aided by substitution. Integration can often be facilitated by the substitution of a new variable for some function of the given independent variable; in other words, by changing the independent variable. Experience is the best guide as to what substitution is likely to transform the given expression into another that is more readily integrable. The advantage of such change or substitution has been made manifest in working some of the examples in Art. 104, *e.g.* Exs. 5, 6, 7, 8, etc.

EXAMPLES.

1. $\int (x+a)^n\,dx$, in which n is *any* constant, excepting -1.

Put $x+a=z$; then $dx=dz$, and

$$\int (x+a)^n\,dx = \int z^n\,dz = \frac{z^{n+1}}{n+1}+c = \frac{(x+a)^{n+1}}{n+1}+c.$$

This may be integrated without explicitly changing the variable. For, since

$$dx = d(x+a),\ \int (x+a)^n\,dx = \int (x+a)^n\,d(x+a) = \frac{(x+a)^{n+1}}{n+1}+c.$$

2. $\int (x+a)^{-1}\,dx = \int \frac{dx}{x+a} = \int \frac{d(x+a)}{x+a} = \log(x+a)+c.$

3. $\int \frac{dx}{x\sqrt{4+3x}}$.

Put $4+3x=z^2$; then $x=\frac{1}{3}(z^2-4)$, and $dx=\frac{2}{3}z\,dz$. Hence, on denoting the integral by I,

$$I=2\int\frac{dz}{z^2-4}=\frac{1}{2}\int\left(\frac{1}{z-2}-\frac{1}{z+2}\right)dz$$

$$=\tfrac{1}{2}\log\frac{z-2}{z+2}+c=\tfrac{1}{2}\log\frac{\sqrt{4+3x}-2}{\sqrt{4+3x}+2}+c.$$

4. $\int\frac{dx}{\sqrt{a^2-x^2}}$.

Put $x=a\sin\theta$. Then $dx=a\cos\theta\,d\theta$, and

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\cos\theta\,d\theta}{\sqrt{a^2-a^2\sin^2\theta}}=\int d\theta=\theta+c=\sin^{-1}\frac{x}{a}+c.$$

This integral may be found by another substitution. For, put $x=az$; then

$$dx=a\,dz,\text{ and }\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\,dz}{\sqrt{a^2-a^2z^2}}=\int\frac{dz}{\sqrt{1-z^2}}$$

$$=\sin^{-1}z+c=\sin^{-1}\frac{x}{a}+c.$$

5. $\int\sqrt{a^2-x^2}\,dx.$

Put $x=a\sin\theta$. Then $dx=a\cos\theta\,d\theta$; and

$$\int\sqrt{a^2-x^2}\,dx=\int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta=a^2\int\cos^2\theta\,d\theta=\frac{a^2}{2}\int(1+\cos2\theta)d\theta$$

$$=\frac{a^2}{2}\left(\theta+\frac{\sin2\theta}{2}\right)+c=\frac{a^2}{2}(\theta+\sin\theta\cos\theta)+c$$

$$=\frac{a^2}{2}\left(\sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right)+c=\tfrac{1}{2}\left(a^2\sin^{-1}\frac{x}{a}+x\sqrt{a^2-x^2}\right)+c.$$

This important integral may also be obtained in other ways; see Ex. 4, Art. 118, and Ex. 5, Art. 106.

6. $\int\frac{du}{a^2+u^2}$. (Put $u=az$.) *Ans.* $\frac{1}{a}\tan^{-1}\frac{u}{a}+c$.

7. $\int\frac{du}{u\sqrt{u^2-a^2}}$. (Put $u=az$.) *Ans.* $\frac{1}{a}\sec^{-1}\frac{u}{a}+c$.

8. $\int\frac{du}{\sqrt{2au-u^2}}$. (Put $u=az$.) *Ans.* $\text{vers}^{-1}\frac{u}{a}+c$.

9. $\int\frac{x\,dx}{\sqrt{x+1}}$.

Put $\sqrt{x+1}=z$. Then $x+1=z^2$, $dx=2z\,dz$, and $\int\frac{x\,dx}{\sqrt{x+1}}=\int\frac{(z^2-1)2z\,dz}{z}$

$$=2\int(z^2-1)\,dz=\tfrac{2}{3}z(z^2-3)+c=\tfrac{2}{3}(x-2)\sqrt{x+1}+c.$$

10. $\int \frac{\cos^3 dx}{\sqrt[3]{\sin x}} dx.$

Put $\sin x = t$. Then $\cos x\, dx = dt$, $\cos^3 x\, dx = \cos^2 x \cdot \cos x\, dx = (1 - t^2)\, dt$.

$$\therefore \int \frac{\cos^3 x\, dx}{\sqrt[3]{\sin x}} = \int \frac{1-t^2}{t^{\frac{1}{3}}} dt = \int (t^{-\frac{1}{3}} - t^{\frac{5}{3}})\, dt = \tfrac{3}{2} t^{\frac{2}{3}} - \tfrac{3}{8} t^{\frac{8}{3}} + c$$

$$= \tfrac{3}{8} t^{\frac{2}{3}}(4 - t^2) + c = \tfrac{3}{8} \sin^{\frac{2}{3}} x (4 - \sin^2 x).$$

11. $\int \sin^5 x \cos x\, dx$, $\int \tan^3 x \sec^4 x\, dx$, $\int \sec^2 (4 - 7x)\, dx$, $\int e^{-2x} dx.$

12. $\int \frac{x^2\, dx}{(x+1)^3}$, $\int \frac{(x+1)^3}{x^2} dx$, $\int \frac{x-2}{\sqrt[3]{x+2}} dx$, $\int x(x-2)^{\frac{1}{3}} dx.$

13. $\int \sqrt[3]{(x+a)^2}\, dx$, $\int \sqrt[5]{(m+nx)^3}\, dx$, $\int \frac{dx}{\sqrt{3-7x}}$, $\int \frac{dy}{\sqrt[5]{(4+5y)^3}}.$

14. $\int e^{m+nx} dx$, $\int 4^{5-3x} dx$, $\int \frac{dx}{(1+x^2)\tan^{-1} x}$, $\int \frac{\sin(\log x)}{x} dx.$

15. $\int t(t-1)^{\frac{1}{2}} dt$, $\int (a+by)^{\frac{2}{3}} dy$, $\int (m+z)^{\frac{3}{4}} dz$, $\int \cos \tfrac{2}{5} x\, dx.$

16. $\int \cos^3 x\, dx$, $\int \sec^4 x\, dx$, $\int \sin^5 x\, dx$, $\int \sec^2 \left(\frac{\theta}{n}\right) d\theta.$

17. $\int \frac{\sin x\, dx}{3 + 7\cos x}$, $\int \frac{\cos x\, dx}{9 - 2\sin x}$, $\int \frac{\sec^2 x\, dx}{\sqrt{4 - 3\tan x}}$, $\int \frac{\sec^2 x\, dx}{\sqrt{10 - 3\sec^2 x}}.$

18. $\int \frac{x\, dx}{\sqrt{a^2 + x^2}}$, $\int (a^2 - x^2)^{\frac{3}{2}} x\, dx$, $\int \sqrt{(a^2 + x^2)} \cdot x\, dx$, $\int \frac{x\, dx}{(a^2 - x^2)^{\frac{3}{2}}}.$

106. Integration by parts. Let u and v denote functions of a variable, say x; then [Art. 32 (7)]

$$d(uv) = u\, dv + v\, du,$$

whence
$$u\, dv = d(uv) - v\, du.$$

Hence, on integration of both members,

$$\int u\, dv = uv - \int v\, du. \tag{1}$$

If an expression $f(x)\, dx$ is not readily integrable, it may be divided into two factors, u and dv say. The application of formula (1) will lead to the integral $\int v\, du$, and it may happen that this integral can easily be found.

NOTE 1. The method of integrating by the application of formula (1) is called *integration by parts. This is one of the most important of the particular methods of integration.*

EXAMPLES.

1. Find $\int xe^x\,dx$.

Put $u = x$; then $dv = e^x\,dx$,

$du = dx$, and $v = e^x$.

$$\therefore \int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + c.$$

2. Find $\int \sin^{-1} x\,dx$.

Put $u = \sin^{-1} x$; then $dv = dx$,

$du = \dfrac{dx}{\sqrt{1-x^2}}$, and $v = x$.

$$\therefore \int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + c.$$ (See Ex. 18, Art. 105.)

3. Find $\int x\cos x\,dx$.

Put $u = \cos x$; then $dv = x\,dx$,

$du = -\sin x\,dx$, and $v = \frac{1}{2}x^2$.

$$\therefore \int x\cos x\,dx = \tfrac{1}{2}x^2\cos x + \tfrac{1}{2}\int x^2 \sin x\,dx.$$

Here the integral in the second member is not as simple a form, from the point of view of integration, as the given form in the first member. Accordingly, it is necessary to try another choice of the factors u and dv.

Put $u = x$; then $dv = \cos x\,dx$,

$du = dx$, and $v = \sin x$.

$$\therefore \int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c.$$

4. Find $\int x^3 \cos x\,dx$.

Put $u = x^3$; then $dv = \cos x\,dx$,

$du = 3x^2\,dx$, and $v = \sin x$.

$$\therefore \int x^3\cos x\,dx = x^3 \sin x - 3\int x^2 \sin x\,dx. \qquad (1)$$

It is now necessary to find $\int x^2 \sin x\,dx$.

Put $u = x^2$; then $dv = \sin x\,dx$,

$du = 2x\,dx$, and $v = -\cos x$.

$$\therefore \int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx. \qquad (2)$$

It is now necessary to find $\int x \cos x \, dx$.

By Ex. 3, $\int x \cos x \, dx = x \sin x + \cos x + c.$

Substitution of this result in (2), and then substitution of result (2) in (1), gives

$$\int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c_1.$$

When the operation of integrating by parts has to be performed several times in succession, *neatness in arranging work* is a great aid in preventing mistakes. The work above may be arranged much more neatly; thus:

$$\begin{aligned}\int x^3 \cos x \, dx &= x^3 \sin x - 3\int x^2 \sin x \, dx \\ &= x^3 \sin x - 3\left[-x^2 \cos x + 2\int x \cos x \, dx\right] \\ &= x^3 \sin x - 3[-x^2 \cos x + 2(x \sin x + \cos x + c)] \\ &= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C \\ &= x(x^2 - 6) \sin x + 3(x^2 - 2) \cos x + C.\end{aligned}$$

The subsidiary work may be kept in another place.

5. Find $\int \sqrt{a^2 - x^2} \, dx$. (See Ex. 5, Art. 105.)

Put $u = \sqrt{a^2 - x^2}$; then $dv = dx$,

$du = -\dfrac{x \, dx}{\sqrt{a^2 - x^2}}$, and $v = x$.

$$\therefore \int \sqrt{a^2 - x^2} \, dx = x\sqrt{a^2 - x^2} + \int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}}. \qquad (1)$$

Now
$$\sqrt{a^2 - x^2} = \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = \frac{a^2}{\sqrt{a^2 - x^2}} - \frac{x^2}{\sqrt{a^2 - x^2}};$$

hence
$$\frac{x^2}{\sqrt{a^2 - x^2}} = \frac{a^2}{\sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}.$$

Substitution in (1) gives

$$\int \sqrt{a^2 - x^2} \, dx = x\sqrt{a^2 - x^2} + \int \frac{a^2 \, dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2} \, dx. \qquad (2)$$

Hence, on transposition of the last integral in (2) to the first member, division by 2, and Ex. 4, Art. 105,

$$\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}\right).$$

6. $\int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x).$

(Integrate, putting $u = e^x$; then integrate, putting $u = \cos x$. Take half the sum of the two results.)

7. $\int xe^{ax}\, dx.$

8. $\int xe^{-x}\, dx.$

9. $\int x^2 e^{\frac{x}{a}}\, dx.$

10. $\int \log x\, dx.$

11. $\int x \log x\, dx.$

12. $\int x^2 \log x\, dx.$

13. $\int \tan^{-1} x\, dx.$

14. $\int x \tan^{-1} x\, dx.$

15. $\int x^2 \sin x\, dx.$

16. $\int e^x x^m\, dx.$

17. $\int x \sin x \cos x\, dx.$

18. $\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}}\, dx.$

19. Derive $\int e^x \sin x\, dx = \frac{1}{2} e^x (\sin x - \cos x).$ (See Ex. 6.)

107. Further elementary integrals. A further list of elementary integrals is given here. They can be verified by differentiation. Some of the ways in which they may be derived are indicated in the latter part of the article.

$$\text{XV.} \quad \int \tan u\, du = \log \sec u + c.$$

$$\text{XVI.} \quad \int \cot u\, du = \log \sin u + c.$$

$$\text{XVII.} \quad \int \sec u\, du = \log (\sec u + \tan u) + c, \\ = \log \tan \left(\frac{u}{2} + \frac{\pi}{4}\right) + c.$$

$$\text{XVIII.} \quad \int \operatorname{cosec} u\, du = \log \tan \frac{u}{2} + c.$$

$$\text{XIX.} \quad \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c.$$

$$\text{XX.} \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c.$$

$$\text{XXI.} \quad \int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + c.$$

$$\text{XXII.} \quad \int \frac{du}{\sqrt{2au - u^2}} = \operatorname{vers}^{-1} \frac{u}{a} + c.$$

N.B. *See Note* 1.

XXIII. $\displaystyle\int\frac{du}{u^2-a^2}=\frac{1}{2a}\log\frac{u-a}{u+a};\ \int\frac{du}{a^2-u^2}=\frac{1}{2a}\log\frac{a+u}{a-u}.$

XXIV. $$\int\frac{du}{\sqrt{u^2+a^2}}=\log(u+\sqrt{u^2+a^2})+c,$$
$$=\log\frac{u+\sqrt{u^2+a^2}}{a}+c'.$$

XXV. $$\int\frac{du}{\sqrt{u^2-a^2}}=\log(u+\sqrt{u^2-a^2})+c,$$
$$=\log\frac{u+\sqrt{u^2-a^2}}{a}+c'.$$

XXVI. $$\int\sqrt{a^2-u^2}\,du=\frac{1}{2}\left(u\sqrt{a^2-u^2}+a^2\sin^{-1}\frac{u}{a}\right)+c.$$

Integral XXII. is also reducible to form XIX. For $2au-u^2 = a^2-(u-a)^2$, and $du=d(u-a)$;

$$\therefore\int\frac{du}{\sqrt{2au-u^2}}=\int\frac{d(u-a)}{\sqrt{a^2-(u-a)^2}}=\sin^{-1}\frac{u-a}{a}+c'.$$

Ex. Show that this result and that in XXII. are equivalent.

Remarks on integrals XV. to XXVI.

Formulas XV., XVI. For derivation, see Exs. 6, 7, Art. 104.
Formulas XVII., XVIII.

Since $$\operatorname{cosec} u=\operatorname{cosec} u\,\frac{\operatorname{cosec} u-\cot u}{\operatorname{cosec} u-\cot u},$$

$$\int\operatorname{cosec} u\,du=\int\frac{-\operatorname{cosec} u\cot u+\operatorname{cosec}^2 u}{\operatorname{cosec} u-\cot u}du$$
$$=\int\frac{d(\operatorname{cosec} u-\cot u)}{\operatorname{cosec} u-\cot u}=\log(\operatorname{cosec} u-\cot u)$$
$$=\log\frac{1-\cos u}{\sin u}=\log\frac{2\sin^2\frac{u}{2}}{2\sin\frac{u}{2}\cos\frac{u}{2}}=\log\tan\frac{u}{2}.$$

Substitution of $u+\frac{\pi}{2}$ for u in the last two lines gives

$$\int\operatorname{cosec}\left(u+\frac{\pi}{2}\right)du=\log\tan\left(\frac{u}{2}+\frac{\pi}{4}\right),\ i.e.\ \int\sec u\,du=\log\tan\left(\frac{u}{2}+\frac{\pi}{4}\right);$$
$$=\log\left\{\operatorname{cosec}\left(u+\frac{\pi}{2}\right)-\cot\left(u+\frac{\pi}{2}\right)\right\}=\log(\sec u+\tan u).$$

There are various methods of deriving XVII. and XVIII.

Formulas XIX., XX., XXI., XXII., XXIII. For derivation, see Exs. 4, 6, 7, 8, Art. 105, and the following suggestion:

SUGGESTION: $\frac{1}{u^2-a^2}=\frac{1}{2a}\left(\frac{1}{u-a}-\frac{1}{u+a}\right)$; $\frac{1}{a^2-u^2}=\frac{1}{2a}\left(\frac{1}{a+u}+\frac{1}{a-u}\right)$.

Formula XXIV.

Put $u^2+a^2=z^2$; then $u\,du=z\,dz$, whence $\frac{du}{z}=\frac{dz}{u}$.

Hence, $$\frac{du}{\sqrt{u^2+a^2}}=\frac{du}{z}=\frac{dz}{u}.$$

On composition, $$\frac{du}{\sqrt{u^2+a^2}}=\frac{du+dz}{u+z}=\frac{d(u+z)}{u+z}.$$

$$\therefore \int\frac{du}{\sqrt{u^2+a^2}}=\int\frac{d(u+z)}{u+z}=\log(u+z)+c=\log(u+\sqrt{u^2+a^2})+c.$$

The last result may be written

$$\log(u+\sqrt{u^2+a^2})-\log a+c',\ \textit{i.e.}\ \log\frac{u+\sqrt{u^2+a^2}}{a}+c',$$

a form which is convenient for some purposes. *See Note* 3.

Formula XXV. can be derived in the same way as XXIV.

Formula XXVI. For derivation, see Ex. 5, Art. 105, and Ex. 5, Art. 106.

Note 1. Integrals XIX., XX., XXI., XXII., may be respectively written

$$-\cos^{-1}\frac{u}{a}+c',\quad -\frac{1}{a}\cot^{-1}\frac{u}{a}+c',\quad -\frac{1}{a}\csc^{-1}\frac{u}{a}+c',\quad -\text{covers}^{-1}\frac{u}{a}+c'.$$

Ex. Show this.

Note 2. Integrals XXIII., XXIV., XXV., may be written thus:

$$\int\frac{du}{a^2-u^2}=\frac{1}{a}\,\text{hy}\tan^{-1}\frac{u}{a}+c'(u^2<a^2),$$

$$\int\frac{du}{u^2-a^2}=-\frac{1}{a}\,\text{hy}\cot^{-1}\frac{u}{a}+c'(u^2>a^2),$$

$$\int\frac{du}{\sqrt{u^2+a^2}}=\text{hy}\sin^{-1}\frac{u}{a}+c',$$

$$\int\frac{du}{\sqrt{u^2-a^2}}=\pm\,\text{hy}\cos^{-1}\frac{u}{a}+c'.$$

The functions whose symbols are here indicated are the inverse hyperbolic tangent of $\frac{u}{a}$, the inverse hyperbolic sine of $\frac{u}{a}$, and the inverse hyperbolic cosine of $\frac{u}{a}$. For a note on **hyperbolic functions** see Appendix, Note A. The close similarity between XX. and these forms of XXIII. may be remarked; so also, between the forms of XIX. and these forms of XXIV. and XXV.

Note 3. *The same integral may be obtained by various substitutions, and may be expressed in a variety of forms.* Instances of this have already been given; another example is the following: Integral XXIV. can also be derived by changing the variable from u to z by means of the substitution $\sqrt{u^2 + a^2} = z - u$; this leads to the form

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \log (u + \sqrt{u^2 + a^2}) + c.$$

The first member can also be integrated by changing the integral from u to z by means of the substitution $\sqrt{u^2 + a^2} = zu$; this leads to the form

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \log \left\{ \frac{\sqrt{u^2 + a^2} + u}{\sqrt{u^2 + a^2} - u} \right\}^{\frac{1}{2}} + c.$$

It is left as *an exercise for the student,* to employ these substitutions in the integration of XXIV., and, the arbitrary constants of integration being excepted, to show the identity of the various forms obtained for the integral.

EXAMPLES.

1. $\int \frac{4 + 7x}{4 + x^2} dx = \int \left(\frac{4}{4 + x^2} + \frac{7x}{4 + x^2}\right) dx = 2 \tan^{-1} \frac{x}{2} + \frac{7}{2} \log (4 + x^2) + c.$

2. $\int \frac{4+7x}{\sqrt{4-x^2}} dx = \int \left(\frac{4}{\sqrt{4-x^2}} + \frac{7x}{\sqrt{4-x^2}}\right) dx = 4 \sin^{-1} \frac{x}{2} - 7(4-x^2)^{\frac{1}{2}} + c.$

3. $\int \frac{dx}{x^2 + 4x + 20} = \int \frac{d(x+2)}{(x+2)^2 + 16} = \frac{1}{4} \tan^{-1} \frac{x+2}{4} + c.$

4 *a*. $\int \frac{dx}{\sqrt{x^2+4x+20}} = \int \frac{d(x+2)}{\sqrt{(x+2)^2 + 16}} = \log(x+2+\sqrt{x^2+4x+20}) + c.$

4 *b*. $\int \frac{dx}{\sqrt{12 - x^2 - 4x}} = \int \frac{d(x+2)}{\sqrt{16 - (x+2)^2}} = \sin^{-1} \frac{x+2}{4} + c.$

Notice should be taken of the aid afforded (*e.g.* in Exs. 3, 4 *a*, 4 *b*) by completing a square involving the terms in x.

5. $\int \frac{dx}{7x\sqrt{4x^2 - 9}} = \frac{1}{7} \int \frac{d(2x)}{2x\sqrt{(2x)^2 - 3^2}} = \frac{1}{21} \sec^{-1} \frac{2x}{3} + c.$

6. $\int \frac{dx}{x^2\sqrt{16-x^2}}$.

Put $x = \frac{1}{t}$. Then $dx = -\frac{1}{t^2}dt$, and

$$\int \frac{dx}{x^2\sqrt{16-x^2}} = -\int \frac{t\,dt}{\sqrt{16t^2-1}} = -\frac{1}{32}\int (16t^2-1)^{-\frac{1}{2}}d(16t^2-1)$$

$$= -\frac{1}{16}(16t^2-1)^{\frac{1}{2}} + c = -\frac{(16-x^2)^{\frac{1}{2}}}{16x} + c.$$

7. (1) $\int \frac{dx}{x^2+6x+17}$; (2) $\int \frac{dx}{\sqrt{17+6x-x^2}}$; (3) $\int \frac{dx}{\sqrt{x^2+6x+10}}$.

8. (1) $\int \frac{dx}{7-6x-x^2}$; (2) $\int \frac{dx}{\sqrt{7-5x-x^2}}$; (3) $\int \frac{dx}{\sqrt{x^2-5x+7}}$.

9. (1) $\int \frac{dx}{x^2+5x-2}$; (2) $\int \frac{dx}{x^2+5x-9}$; (3) $\int \frac{dx}{\sqrt{4x^2-3x+5}}$.

10. (1) $\int \frac{dx}{4x^2-5x+6}$; (2) $\int \frac{dx}{\sqrt{9-5x-4x^2}}$; (3) $\int \frac{dx}{7-5x-4x^2}$.

11. (1) $\int \frac{dx}{\sqrt{8x-x^2}}$; (2) $\int \frac{dx}{\sqrt{9x-4x^2}}$; (3) $\int \frac{dx}{5x\sqrt{9x^2-25}}$.

12. (1) $\int \frac{dx}{(x-1)\sqrt{x^2-2x-3}}$; (2) $\int \sqrt{9-x^2}\,dx$; (3) $\int_0^5 \sqrt{25-x^2}\,dx$.

13. (1) $\int \sqrt{36-4x^2}\,dx$; (2) $\int \sec 3x\,dx$; (3) $\int \operatorname{cosec}(4x-\alpha)\,dx$.

14. (1) $\int \tan(3x+a)dx$; (2) $\int \cot(4x^2+a^2)x\,dx$; (3) $\int \sec 2x\,dx$.

15. Derive integrals 62 *a*, *b*, 63 *a*, *b*, p. 406.

16. $\int \frac{\sqrt{25-x^2}}{x^4}dx$, $\int \frac{dx}{(4+x^2)^{\frac{3}{2}}}$, $\int \frac{dx}{x\sqrt{12x-x^2}}$.

108. Integration of $f(x)dx$ when $f(x)$ is a rational fraction. In order to find $\int f(x)\,dx$ when $f(x)$ is a rational fraction, the procedure is as follows:

Resolve $f(x)$ into component fractions, and integrate the differentials involving the component fractions.

NOTE. It is here taken for granted that in his course in algebra the student has been made familiar with the decomposition of a rational fraction into component fractions, or, as it is usually termed, the resolution of a rational fraction into partial fractions. Reference may be made to works on algebra, *e.g.* Chrystal, *Algebra*, Part I., Chap. VIII.; also to texts on calculus, *e.g.* Snyder and Hutchinson, *Calculus*, Arts. 132–137.

Examples 1, 2, 4 will serve to recall to mind the practical points that are necessary for present purposes.

EXAMPLES.

1. $\displaystyle\int \frac{x^3 - 3x^2 + 4x + 14}{x^2 + x - 6}\,dx.$

Here $$\frac{x^3 - 3x^2 + 4x + 14}{x^2 + x - 6} = x - 4 + \frac{14x - 10}{x^2 + x - 6}.$$

The fraction in the second member is *a proper fraction*, and is *in its lowest terms*. Accordingly, the work of resolving it into fractions having denominators of lower degree than the second, may be proceeded with. Since its denominator, $x^2 + x - 6$, *i.e.* $(x - 2)(x + 3)$, is the common denominator of the component fractions, one of the latter evidently must have a denominator $x - 2$, and the other a denominator $x + 3$. Since these fractions must be proper fractions, their numerators must be of lower degrees than the denominators, and, accordingly, must be constants.

Accordingly, put

$$\frac{14x - 10}{x^2 + x - 6} = \left(\frac{14x - 10}{(x - 2)(x + 3)}\right) = \frac{A}{x - 2} + \frac{B}{x + 3}. \tag{1}$$

Here A and B are to be determined so that the two members of (1) shall be identically equal.

On clearing of fractions,

$$14x - 10 = A(x + 3) + B(x - 2). \tag{2}$$

Since the members of (2) are to be identically equal, the coefficients of like powers of x must be equal. That is,

$$A + B = 14,$$
$$3A - 2B = -10.$$

On solving these equations, $A = \frac{18}{5}$, $B = \frac{52}{5}$.

$$\therefore \int \frac{x^3 - 3x^2 + 4x + 14}{x^2 + x - 6}\,dx = \int \left(x - 4 + \frac{18}{5(x - 2)} + \frac{52}{5(x + 3)}\right) dx$$
$$= \tfrac{1}{2}x^2 - 4x + \tfrac{18}{5}\log(x - 2) + \tfrac{52}{5}\log(x + 3) + c.$$

Another way of finding A and B in (2) is the following:

The two members of (2) are to be identically equal, and accordingly equal for all values of x.

Now, put $x = -3$; then $-5B = -52$; whence, $B = \frac{52}{5}$.

Put $x = 2$; then $5A = 18$; whence, $A = \frac{18}{5}$.

NOTE 1. Any other values, *e.g.* 3 and 7, may be assigned to x; in this case, however, the values 2 and -3 give the most convenient equations for determining A and B.

NOTE 2. For a more rapid way of finding A and B in such cases as (1), see Murray, *Integral Calculus*, Appendix, Note A.

2. $\int \frac{x^2+21x-10}{x^3+x^2-5x+3}\,dx.$

The fraction in the integrand is a proper fraction, and is in its lowest terms. Accordingly, the work of decomposing it into fractions having denominators of degrees lower than the third may be proceeded with. Since the denominator x^3+x^2-5x+3, *i.e.* $(x-1)^2(x+3)$ is the common denominator of the component fractions, one of the latter evidently must have a denominator $x+3$, and another must have a denominator $(x-1)^2$. It is also possible that there may be a component fraction having the denominator $x-1$; for, if there is such a fraction, it does not affect the given common denominator. Accordingly, put

$$\frac{x^2+21x-10}{(x-1)^2(x+3)}=\frac{A}{x+3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}, \tag{3}$$

in which A, B, C are constants to be determined.

On clearing of fractions, equating like powers of x (for reasons indicated in Ex. 1), and solving for A, B, C, it is found that

$$A=-4, \quad B=3, \quad C=5.$$

$$\therefore \int \frac{x^2+21x-10}{x^3+x^2-5x+3}\,dx=\int\left(\frac{-4}{x+3}+\frac{3}{(x-1)^2}+\frac{5}{x-1}\right)dx$$

$$=5\log(x-1)-4\log(x+3)-\frac{3}{x-1}+c=\log\frac{(x-1)^5}{(x+3)^4}-\frac{3}{x-1}+c.$$

NOTE 3. It may be asked why the numerator assigned to the quadratic denominator $(x-1)^2$ in the second member of (3) is not an expression of the first degree in x, say $Bx+D$, instead of a constant. The reason is, that if such a numerator were assigned, the fraction would immediately reduce to the forms in (3). For

$$\frac{Bx+D}{(x-1)^2}=\frac{B(x-1)+D+B}{(x-1)^2}=\frac{B}{x-1}+\frac{D+B}{(x-1)^2},$$

forms which appear in (3).

NOTE 4. If a factor of the form $(x-a)^r$ appears among the factors of the denominator of the fraction to be resolved, there evidently must be a component fraction having $(x-a)^r$ for its denominator. There may also possibly be fractions having as denominators $(x-a)$ of various powers less than r, *e.g.* $(x-a)^{r-1}$, $(x-a)^{r-2}$, $\cdots$, $x-a$. Accordingly, in such a case it is necessary to allow also for the possibility of the existence of fractions of the forms

$$\frac{M}{(x-a)^{r-1}},\ \frac{F}{(x-a)^{r-2}},\ \cdots,\ \frac{L}{x-a},$$

in which M, F, $\cdots$, L, are constants.

3. $\int \frac{2x^2 - 8x - 10}{x^3 + x^2 - 5x + 3} dx.$ (Compare denominators in Exs. 2, 3.)

4. $\int \frac{5x^2 + 3x + 17}{x^3 - x^2 + 4x - 4} dx.$

The fraction in the integrand is a proper fraction and is in its lowest terms. If it were not so, division as in Ex. 1 and reduction would be necessary. Since the denominator $x^3 - x^2 + 4x - 4$, *i.e.* $(x^2 + 4)(x - 1)$, is the common denominator of the component fractions, one of the latter must have a denominator $x^2 + 4$, and the other a denominator $x - 1$. Accordingly, put

$$\frac{5x^2 + 3x + 17}{(x^2 + 4)(x - 1)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x - 1},$$

in which A, B, C, are constants to be determined.

On clearing of fractions, equating coefficients of like powers of x, and solving for A, B, C, it is found that

$$A = 0, \; B = 3, \; C = 5.$$

$$\therefore \int \frac{5x^2 + 3x + 17}{x^3 - x^2 + 4x - 4} dx = \int \left(\frac{3}{x^2 + 4} + \frac{5}{x - 1} \right) dx$$

$$= \frac{3}{2} \tan^{-1} \frac{x}{2} + 5 \log (x - 1) + c.$$

NOTE 5. The expression $x^2 + 4$ has factors $x + 2i$, $x - 2i$ $(i = \sqrt{-1})$; if these be taken, component fractions imaginary in form, are obtained. It is usual, however, not to carry the decomposition of a fraction as far as the stage in which component fractions imaginary in form may appear.

NOTE 6. The numerator $Ax + B$ is assigned above; for the numerator over a *quadratic* denominator whose factors are imaginary, may have the form of the most general expression of *the first degree* in x.

NOTE 7. When a quadratic expression $x^2 + px + q$ has imaginary factors and is repeated r times in the denominator of a fraction, in the process of decomposition of this fraction allowance must be made for fractions of the

forms, $$\frac{Ax + B}{(x^2 + px + q)^r}, \; \frac{Cx + D}{(x^2 + px + q)^{r-1}}, \; \ldots, \; \frac{Mx + N}{x^2 + px + q}.$$

5. (1) $\int \frac{11x^2 - 4x + 28}{x^3 - x^2 + 4x - 4} dx$; (2) $\int \frac{3x^2 - 13x - 5}{x^3 - x^2 + 4x - 4} dx.$ (Compare the denominators in Exs. 4, 5.)

Find the anti-derivatives of the following fractions:

6. $\dfrac{x+37}{x^2-3x-28}$.

7. $\dfrac{8x+1}{2x^2-9x-35}$.

8. $\dfrac{x^3-2x^2-1}{x^2-1}$.

9. $\dfrac{x^4-x^2+1}{x^3-x}$.

10. $\dfrac{x^2-10x-5}{x(2x^2+3x-5)}$.

11. $\dfrac{x^2+pq}{x(x-p)(x+q)}$.

12. $\dfrac{11x^3-11x^2-74x+84}{x^4-13x^2+36}$.

13. $\dfrac{x+1}{(x-1)^2}$.

14. $\dfrac{8x+5}{(4x+5)^2}$.

15. $\dfrac{5x^2+x-10}{x^2(2x+5)}$.

16. $\dfrac{30x^2+43x-8}{(x+4)(3x+2)^2}$.

17. $\dfrac{2x^2}{(x+1)^3}$.

18. $\dfrac{x^2-3x+3}{x(x^2+3)}$.

19. $\dfrac{12-x-x^2}{(3x-2)(x^2+5)}$.

20. $\dfrac{(x+1)^2}{x^3+x}$.

21. $\dfrac{x^3-1}{x^3+3x}$.

22. $\dfrac{2x^2+3x+6}{x^3+3x}$.

23. $\dfrac{7x^2+9}{x^3+3x}$.

24. $\dfrac{2x^3-x^2+8x+12}{x^2(x^2+4)}$.

25. $\dfrac{2+3x-x^2}{(x-1)(x^2-2x+5)}$.

26. $\dfrac{1+7x+x^2+x^3}{(x^2+1)^2}$.

Ex. 27. Show that any expression of the form $\displaystyle\int\frac{(mx+n)\,dx}{ax^2+bx+c}$, in which m, n, a, b, and c are constants, is integrable.

109. Integration of a total differential. In Art. 86 it has been shown that the *necessary* condition for the existence of a function having

$$P\,dx+Q\,dy \tag{1}$$

for its differential, is that

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}. \tag{2}$$

It has also been stated (Art. 86, Note 1) that condition (2) is *sufficient* for the existence of such a function. In other words, if the expression (1) has an anti-differential (or integral), relation (2) must be satisfied; conversely, if relation (2) is satisfied, the expression (1) has an integral. Accordingly, relation (2) is called *the criterion of integrability* for the expression (1). If this criterion

is satisfied, the expression (1) is said to be *a complete differential*, *a total differential*, and also *an exact differential.*

If test (2) is satisfied, the integral of (1) can easily be found. This integral's partial x-differential, $P\,dx$, can only come from terms containing x (Art. 79). Hence, the integral of $P\,dx$ with respect to x, namely,

$$\int P\,dx + c, \tag{3}$$

must yield all the terms of the required integral that contain x. Also, $Q\,dy$ can only come from terms containing y. Hence the integral of $Q\,dy$ with respect to y, namely,

$$\int Q\,dy + c_2 \tag{4}$$

must yield all the terms of the required integral that contain y. Some of these terms may contain x; if so, they have already been obtained in (3), and need not be taken this second time. Hence, *if the integral of a differential of the form*

$$P\,dx + Q\,dy$$

is required, *apply the test for integrability, namely,*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x};$$

if this test is satisfied, integrate $P\,dx$ with respect to x; then integrate $Q\,dy$ with respect to y, neglecting terms already obtained in $\int P\,dx$; add the results and the arbitrary constant of integration.

EXAMPLES.

1. Integrate $(2\,xy + 2 + 3\,y^2 + 12\,x)\,dx + (x^2 + 6\,xy + 4\,y^3)\,dy$.

Here $\quad P = 2\,xy + 2 + 3\,y^2 + 12\,x$, and $\quad Q = x^2 + 6\,xy + 4\,y^3$.

$$\therefore\ \frac{\partial P}{\partial y} = 2\,x + 6\,y, \qquad \text{and } \frac{\partial Q}{\partial x} = 2\,x + 6\,y.$$

Thus the criterion of integrability is satisfied.

Also $\int P\,dx = x^2y + 2\,x + 3\,xy^2 + 6\,x^2$;

and $\quad \int Q\,dy = x^2y + 3\,xy^2 + y^4$, in which y^4 has not been already obtained in $\int P\,dx$. Hence the integral is

$$x^2y + 2\,x + 3\,y^2 + 6\,x^2 + y^4 + c.$$

2. Verify the result in Ex. 1 by differentiation.

3. Find $\int (x\,dy - y\,dx)$.

Here $\frac{\partial Q}{\partial x} = 1$, and $\frac{\partial P}{\partial y} = -1$; hence the test for integrability is not satisfied, and there is not an anti-differential.

4. (1) $\int e^x (\cos y\,dx - \sin y\,dy)$. (2) $\int [(3x^2 + 8xy + 4)dx + (4x^2 - 6)dy]$.

5. Integrate: (1) $\cos x \sec^2 y\,dy - (\sin x \tan y + \cos x)\,dx$.
(2) $(xe^y - 2x)\,dy + (e^y - 2y + 2x)\,dx$. (3) $(3 - 4x - y)\,dx - (x + y)\,dy$.

N.B. An accurate and ready memory of the fundamental integrals (Arts. 103, 107), resourcefulness in making substitutions (Art. 105), and quickness in integrating by parts (Art. 106), are three very important things to cultivate in order to insure comfortable progress in the study of the calculus.

EXAMPLES.

1. $\int ln^2 x^{\sqrt{2}+m}\,dx$, $\int (a+b)x^{2(a+b)-1}\,dx$, $\int (r+s)z^{n+t+2}\,dz$, $\int r^{\frac{2}{3}}s^{\frac{1}{2}}y^{rs-1}\,dy$, $\int_0^1 \frac{t^4 - 6t + 1}{t+2}\,dt$, $\int \frac{v^3 + 8v^2 - 9}{v^2 + 3}\,dv$, $\int \frac{x^3 - 2x^2 + 7x - 1}{x^2 - 2}\,dx$, $\int \frac{7\,dt}{9t^2 + 20}$, $\int \frac{dz}{z^2 - 12}$, $\int_1^8 (a^{\frac{2}{3}} + y^{\frac{2}{3}})^2\,dy$, $\int \frac{x^2\,dx}{\sqrt{9 - x^6}}$, $\int \frac{x^2\,dx}{\sqrt{x^6 - 9}}$, $\int \frac{z^2 - 1}{(2z - 1)^2}\,dz$.

2. $\int \tan(mx + n)\,dx$, $\int (\sec 3x + 2)^2\,dx$, $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \tan 2\theta\,d\theta$, $\int_0^\pi \sin\left(\frac{\theta}{4} + \frac{\pi}{6}\right)d\theta$.

3. $\int \cos^{-1} x\,dx$, $\int \sec^{-1} x\,dx$, $\int \cot^{-1} x\,dx$, $\int (\log x)^2\,dx$, $\int x^2 e^{-\frac{x}{a}}\,dx$, $\int x^3 e^{-x}\,dx$, $\int \sin x \log \cos x$, $\int x^m \log x$.

4. $\int \frac{x^{\frac{1}{3}} - 1}{3x^{\frac{1}{2}}}\,dx$, $\int \frac{3x^{\frac{1}{2}}}{x^{\frac{1}{3}} - 1}\,dx$, $\int_2^3 \frac{dx}{x^{\frac{3}{4}} - x^{\frac{1}{2}}}$, $\int \sqrt{\frac{x+1}{x-1}}\,dx$.

5. $3\int_0^{\frac{\pi}{4}} \tan\frac{\theta}{2}\,d\theta$, $\int_0^1 \frac{dx}{e^{3x}}$, $\int_0^1 e^{3x}\,dx$, $\int_1^{\frac{\sqrt{3}}{2}} \frac{(\sin^{-1} x)^2\,dx}{\sqrt{1 - x^2}}$.

6. $\int \frac{a \sin\theta\,d\theta}{m + n\cos\theta}$, $\int \frac{(1 + \cos\theta)\,d\theta}{\sin\theta}$, $\int \frac{dx}{\sin x + \cos x}$, $\int \frac{\sec x \tan x\,dx}{(\tan^2 x - 3)^2}$, $\int \frac{d\theta}{\cos^2\theta\sqrt{4 - \tan^2\theta}}$, $\int \frac{\log^2 (mz + n)}{mz + n}\,dz$, $\int \frac{dx}{\sqrt{a^{2x} - m^2}}$, $\int \frac{dx}{e^x + e^{-x}}$,

$$\int\frac{dx}{e^{2x}-e^{-2x}},\qquad \int\frac{d\theta}{\cos^2 2\theta-\sin^2 2\theta},\qquad \int\frac{\sin\frac{x}{2}\,dx}{\sin\frac{x}{4}\sqrt{\cos\frac{x}{2}}},$$

$$\int[(1-\sin x\cos y)\,dx-(\cos x\sin y+2y)\,dy],$$

$$\int[(1-\sin x\sin y)\,dx+(\cos x\cos y-1)\,dy].$$

7. Derive the following integrals:

(1) $\int x(x^2\pm a^2)^n\,dx=\frac{(x^2\pm a^2)^{n+1}}{2(n+1)}$. (2) $\int\frac{x\,dx}{\sqrt{x^2\pm a^2}}=\sqrt{x^2\pm a^2}$.

(3) $\int x(a^2-x^2)^n\,dx=-\frac{(a^2-x^2)^{n+1}}{2(n+1)}$. (4) $\int\frac{x\,dx}{\sqrt{a^2-x^2}}=-\sqrt{a^2-x^2}$.

8. Derive the following integrals:

(1) $\int\frac{dx}{a+bx}=\frac{1}{b}\log(a+bx)$. (2) $\int(a+bx)^n\,dx=\frac{(a+bx)^{n+1}}{b(n+1)}$, when n is different from -1. (3) $\int\frac{x\,dx}{a+bx}=\frac{1}{b^2}[a+bx-a\log(a+bx)]$. (4) $\int\frac{x^2\,dx}{a+bx}=\frac{1}{b^3}[\frac{1}{2}(a+bx)^2-2a(a+bx)+a^2\log(a+bx)]$. (5) $\int\frac{dx}{x(a+bx)}=-\frac{1}{a}\log\frac{a+bx}{x}$. (6) $\int\frac{dx}{x^2(a+bx)}=-\frac{1}{ax}+\frac{b}{a^2}\log\frac{a+bx}{x}$. (7) $\int\frac{x\,dx}{(a+bx)^2}=\frac{1}{b^2}\left[\log(a+bx)+\frac{a}{a+bx}\right]$.

9. Derive the following integrals:

(1) $\int\frac{dx}{a^2-b^2x^2}=\frac{1}{2ab}\log\frac{a+bx}{a-bx}$. (2) $\int\frac{dx}{a+bx^2}=\frac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\frac{b}{a}}$, when $a>0$ and $b>0$. (3) $\int\frac{x\,dx}{a+bx^2}=\frac{1}{2b}\log\left(x^2+\frac{a}{b}\right)$. (4) $\int\frac{x^2\,dx}{a+bx^2}=\frac{x}{b}-\frac{a}{b}\int\frac{dx}{a+bx^2}$. (5) $\int\frac{dx}{x(a+bx^2)}=\frac{1}{2a}\log\frac{x^2}{a+bx^2}$. (6) $\int\frac{dx}{x^2(a+bx^2)}=-\frac{1}{ax}-\frac{b}{a}\int\frac{dx}{a+bx^2}$. (7) $\int\frac{x\,dx}{(a+bx^2)^n}=-\frac{1}{2b(n-1)(a+bx^2)^{n-1}}$.

10. Derive the following integrals:

(1) $\int x\sqrt{a+bx}\,dx=-\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}$. (2) $\int x^2\sqrt{a+bx}\,dx=\frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$. (3) $\int\frac{x\,dx}{\sqrt{a+bx}}=-\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}$.

(4) $\int\frac{x^2\,dx}{\sqrt{a+bx}}=\frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}$. (5) $\int\frac{dx}{x\sqrt{a+bx}}=\frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}$, for $a>0$; $\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}$, for $a<0$.

CHAPTER XII.

SIMPLE GEOMETRICAL APPLICATIONS OF INTEGRATION.

110. This chapter treats of some simple geometrical applications of integration. Examples of some of these applications have already appeared in Arts. 96, 97. In Art. 111 integration is used in measuring plane areas, in Art. 112 in measuring the volumes of solids of revolution. In Art. 113 the equations of curves are deduced from given properties whose expression involves derivatives or differentials.

N.B. The student is strongly recommended to draw the figure for each example. In the case of examples which are solved in the text he will find it extremely beneficial to solve, or try to solve, the examples independently of the book.

111. Areas of curves: Cartesian coördinates.

A. **Rectangular axes.** In Art. 96 it has been shown that for a figure bounded by the curve

$$y = f(x),$$

the x-axis, and the two ordinates for which $x = a$ and $x = b$ respectively, the axes being rectangular, area of figure = limit of sum of quantities $y\,\Delta x$ (or $f(x)\,\Delta x$) when Δx approaches zero and x varies continuously from a to b. This limit is denoted by $\int_a^b y\,dx$ or $\int_a^b f(x)\,dx$; it is obtained by finding the anti-differential of $f(x)\,dx$, substituting b and a in turn for x in this anti-differential, and taking the difference between the results of the substitutions. In fewer words: *the number of units in the area is the same as the number of units in a certain definite integral; namely,*

$$\textbf{area of figure} = \int_a^b y\,dx = \int_a^b f(x)\,dx. \qquad (1)$$

The infinitesimal differential $y\,dx$ is called *an element of area.*

N.B. It will be found that in many problems it is necessary:

(1) To find a differential expression for an infinitesimal element of area, or volume, or length, etc., as the case may be.

(2) To reduce this expression to another involving only a single variable.

(3) To integrate the second expression between limits (end-values of the variable), which are either assigned or determinable.

***B.* Oblique axes.** Suppose that the axes are inclined at an angle ω, and that the area of the figure bounded by the curve whose equation is $y = f(x)$, the x-axis, and the ordinates AP and BQ (for which $x = a$ and $x = b$ respectively), is required. Let RM be a parallelogram inscribed between A and B, as rectangles were inscribed in the figures in Arts. 95, 96.

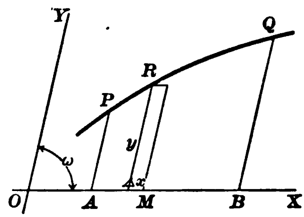

FIG. 47.

$$\text{Area of } RM = y\,\Delta x \cdot \sin\omega.$$

Area $APQB$ = limit of sum of all the parallelograms like RM, infinite in number, that can be inscribed between AP and BQ; that is,

$$\textbf{area } APQB = \int_{x=a}^{x=b} y \sin\omega \cdot dx = \sin\omega \int_a^b y\,dx.$$

Unless otherwise specified, the axes used in the examples in this chapter are rectangular.

EXAMPLES.

1. Find the area between the line $2y - 5x - 7 = 0$, the x-axis, and the ordinates for which $x = 2$ and $x = 5$.

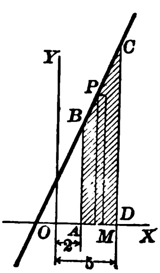

FIG. 48.

The rectangle PM represents an element of area, $y\,dx$. The area required is the limit of the sum of these elementary rectangles, infinite in number, from AB to DC. That is,

$$\text{area} = \int_{x=2}^{x=5} y\,dx = \tfrac{1}{2}\int_2^5 (5x+7)\,dx = \tfrac{1}{2}\left[\frac{5x^2}{2} + 7x\right]_2^5$$

$$= 36\tfrac{3}{4} \text{ square units.}$$

If the unit of length used in drawing the figure were one inch, the figure would contain $36\frac{3}{4}$ square inches.

2. Solve Ex. 1 without the calculus, and thus verify the result obtained by the calculus.

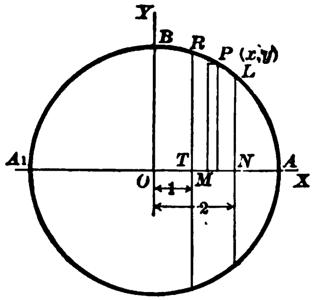

FIG. 49.

3. (*a*) Find the area of the circle $x^2 + y^2 = 9$; (*b*) find the area of the figure bounded by this circle, and the chords for which $x = 1$ and $x = 2$.

Let APB be the circle whose equation is $x^2 + y^2 = 9$. Take a rectangle PM, supposed to be infinitesimal, with a width dx, for the element of area. Its area is $y\,dx$. The area of the quadrant AOB is the limit of the sum of all these elements of area, infinite in number, between O and A. Hence,

$$OAB = \int_{x=0}^{x=3} y\,dx = \int_0^3 \sqrt{9-x^2}\,dx = \tfrac{1}{2}\left[x\sqrt{9-x^2} + 9\sin^{-1}\frac{x}{3}\right]_0^3 = \tfrac{9}{4}\pi \text{ sq. units.}$$

$$\therefore \text{ area circle} = 4 \cdot OAB = 9\pi \text{ square units.}$$

(*b*) Draw the ordinates TR and NL at the points T and N where $x = 1$ and $x = 2$ respectively. The area of $TRLN$ is equal to the limit of the sum of all the elements of area, PM, that lie between TR and NL. That is,

$$\begin{aligned}\text{area } TRLN &= \int_{x=1}^{x=2} y\,dx = \int_1^2 \sqrt{9-x^2}\,dx = \tfrac{1}{2}\left[x\sqrt{9-x^2} + 9\sin^{-1}\frac{x}{3}\right]_1^2 \\ &= \tfrac{1}{2}\{(2\sqrt{5} + 9\sin^{-1}\tfrac{2}{3}) - (\sqrt{8} + 9\sin^{-1}\tfrac{1}{3})\} \\ &= \sqrt{5} - \sqrt{2} + \tfrac{9}{2}(\sin^{-1}\tfrac{2}{3} - \sin^{-1}\tfrac{1}{3}).\end{aligned}$$

Here the *radian measures* of the angles are to be employed.

Now

$\sqrt{2} = 1.414$; $\sin^{-1}\frac{2}{3} = (41^\circ\, 40.8') = .727$ radians; $\sin^{-1}\frac{1}{3} = .340$ radians.

$$\therefore \text{ area required} = 2 \cdot TRLN = 5.126 \text{ square units.}$$

NOTE 1. Other end-values of x may be used in finding the area of this circle. Thus

$$\begin{aligned}\text{area circle} &= 2A_1BA = 2\int_{-3}^{3} y\,dx = 2\int_{-3}^{3}\sqrt{9-x^2}\,dx = \left[x\sqrt{9-x^2} + 9\sin^{-1}\frac{x}{3}\right]_{-3}^{3} \\ &= 9\sin^{-1}1 - 9\sin^{-1}(-1) = \frac{9\pi}{2} - 9\left(-\frac{\pi}{2}\right) = 9\pi \text{ square units.}\end{aligned}$$

NOTE 2. These problems may be stated thus: Find by the calculus (*a*) the area of a circle of radius 3, (*b*) the area of a segment between two parallel chords, distant 1 and 2 units, respectively, from the centre. In this case it is necessary to choose axes (as conveniently as possible), to find the equation of the circle, and then to proceed as above.

4. Find the area between the curve $y = 2x^3$, the y-axis, and the lines $y = 2$ and $y = 4$.

The area is represented by $ABLR$. At any point $P(x, y)$ on the arc RL take for the element of area an infinitesimal rectangle MP. Its area is $x\,dy$.

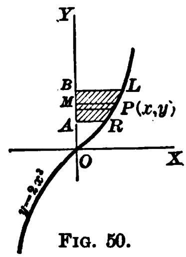

FIG. 50.

$$\therefore \text{ area } ARLB = \int_{y=2}^{y=4} x\,dy = \frac{1}{\sqrt[3]{2}}\int_2^4 y^{\frac{1}{3}}\,dy$$

$$= \frac{1}{2^{\frac{1}{3}}}\left[\frac{3}{4}y^{\frac{4}{3}} + c\right]_2^4 = \frac{1}{2^{\frac{1}{3}}}\cdot\frac{3}{4}\left(4^{\frac{4}{3}} - 2^{\frac{4}{3}}\right)$$

$$= \frac{3}{4}\cdot\frac{1}{2^{\frac{1}{3}}}\cdot 2^{\frac{4}{3}}\left(2^{\frac{4}{3}} - 1\right) = \frac{3}{2}\left(\sqrt[3]{16} - 1\right) = 2.2797.$$

NOTE 3. The definite integral which gives the area may also be expressed in terms of x. For, since $y = 2x^3$, $dy = 6x^2dx$; also, when $y = 2$, $x = 1$, and when $y = 4$, $x = \sqrt[3]{2}$.

$$\therefore \text{ area } ARLB = \int_{y=2}^{y=4} x\,dy = \int_{x=1}^{x=\sqrt[3]{2}} 6x^3dx = \tfrac{3}{2}\left(\sqrt[3]{16} - 1\right) = 2.2797.$$

5. (*a*) Find the area of the figure bounded by the parabola $y^2 = 4ax$, the x-axis, and the ordinate for which $x = x_1$. Show that this area is equal to two-thirds of the rectangle circumscribing the figure. (*b*) Find the area bounded by the parabola $y^2 = 9x$, and the chords for which $x = 4$ and $x = 9$.

6. Find the area between the curve $y^2 = 4x$, the axis of y, and the line whose equation is $y = 6$.

7. Find the area included between the parabolas whose equations are $y^2 = 8x$ and $x^2 = 8y$.

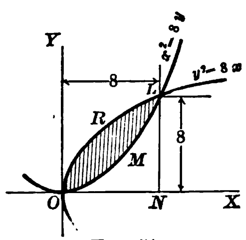

FIG. 51.

The parabolas are OML and ORL; the area of $ORLMO$ is required. To find the points of intersection of the curve, solve these equations simultaneously. This gives (0, 0) the point O, which is otherwise apparent, and (8, 8) the point L.

$$\text{Area } ORLMO = \text{area } ORLN - \text{area } OMLN$$

$$= \sqrt{8}\int_0^8 x^{\frac{1}{2}}\,dx - \tfrac{1}{8}\int_0^8 x^2dx$$

$$= \tfrac{128}{3} - \tfrac{64}{3} = 21\tfrac{1}{3} \text{ square units.}$$

8. Find the area included between the parabolas whose equations are $3y^2 = 25x$ and $5x^2 = 9y$.

9. Find the area included between the parabola $(y - x - 3)^2 = x$, the axes of coördinates, and the line $x = 9$. Figure 52 shows that this problem is ambiguous, for $OTGML$ and $OTKNL$ are each bounded as described. On solving the equation of the curve for y,

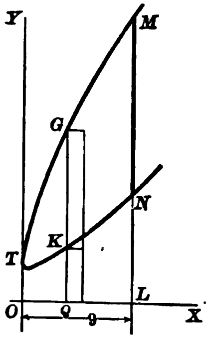

FIG. 52.

$$y = x \pm \sqrt{x} + 3.$$

Thus if $\qquad OQ = x, \;\; QG = x + \sqrt{x} + 3,$

and $\qquad QK = x - \sqrt{x} + 3.$

$\therefore$ area $OTGML$

$$= \int_0^9 (x + \sqrt{x} + 3)\, dx = 85\tfrac{1}{2} \text{ square units;}$$

and area $OTKNL$

$$= \int_0^9 (x - \sqrt{x} + 3)\, dx = 49\tfrac{1}{2} \text{ square units.}$$

Also, the area MTN (the figure bounded by the curve and the chord for which $x = 9$) = area $OTGML$ − area $OTKNL$ = 36 square units.

The area of MTN can also be found as follows:

Area MTN = limit of sum of infinite number of infinitesimal strips, like KG, lying between T and MN.

Now strip $\qquad KG = (QG - QK)\, dx = 2\sqrt{x}\, dx.$

$$\therefore \text{ area } MTN = \int_0^9 2\sqrt{x}\, dx = 36.$$

10. Apply the second method used in finding area MTN in Ex. 9 to finding the areas in Exs. 7 and 8.

11. Find in two ways the area between the parabola $(y - x - 5)^2 = x$ and the chord for which $x = 5$.

12. Find the area between the parabola $y = x^2 - 8x + 12$, the x-axis, and the ordinates at $x = 1$ and $x = 9$.

$$\text{Area} = \int_{x=1}^{x=9} y\, dx = \int_1^9 (x^2 - 8x + 12)\, dx$$

$$= 18\tfrac{2}{3} \text{ square units.} \qquad (1)$$

The parabola crosses the x-axis at B and C where

$$x = 2 \text{ and } x = 6.$$

$$\text{Area } APB = \int_{x=1}^{x=2} y\, dx = 2\tfrac{1}{3};$$

$$\text{area } BEC = \int_2^6 y\, dx = -10\tfrac{2}{3};$$

$$\text{area } CQD = \int_6 y\, dx = 27.$$

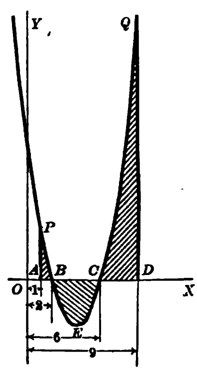

FIG. 53.

$$\text{Area required} = \text{area } APB + \text{area } BEC + \text{area } CQD$$
$$= 2\tfrac{1}{3} - 10\tfrac{2}{3} + 27 = 18\tfrac{2}{3}, \text{ as in (1).}$$

The sign of the area BEC comes out negative, because the element of area, $y\,dx$, is negative as x increases from OB to OC; for dx is then positive and y is negative. On the other hand as x proceeds from A to B and from C to D, $y\,dx$ is positive. The actual area shaded in the figure is $2\tfrac{1}{3} + 10\tfrac{2}{3} + 27$, *i.e.* 40 square units.

N.B. It should be carefully observed, as illustrated in this example, that in the calculus method of finding areas bounded by a curve, the x-axis, and a pair of ordinates, areas above the x-axis come out with a positive, and areas below the x-axis come out with a negative sign. Accordingly, *the calculus gives the algebraic sum of these areas; and this is really the difference between the areas above the x-axis and the areas below it.*

13. (*a*) Find the area bounded by the x-axis and a semi-undulation of the sine curve $y = \sin 2x$. (*b*) Find the area bounded by the x-axis and a complete undulation of the same curve. (*c*) Explain the result *zero* which the calculus gives for (*b*). (*d*) What is the number of square units bounded as in (*b*)?

14. Construct the figure, and show that, according to the calculus method of computing areas, the area between the curve whose equation is $12y = (x-1)(x-3)(x-5)$, the x-axis, and the ordinates for which $x = -2$ and $x = 7$, is $-\frac{99}{16}$ square units; but that the actual number of square units in the figure thus bounded is $12\frac{41}{48}$.

15. Find the area between the line $2y - 5x - 7 = 0$, the x-axis, and the ordinates for which $x = 2$ and $x = 5$, the axes being inclined at an angle 60°.

$$\text{Area } APQB = \int_{x=2}^{x=5} y \sin 60^\circ \cdot dx$$
$$= \sin 60^\circ \int_2^5 (5x + 7)dx$$
$$= 63.65 \text{ square units.}$$

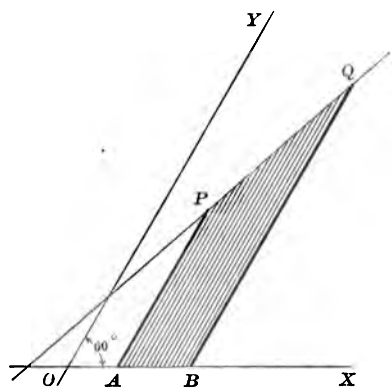

FIG. 54.

Note 4. In the light of the preceding examples attention may be again directed to the N.B. above. These examples also show: (1) the element of area may be chosen in various ways (compare Exs. 1, 4, 7, 9, 11); (2) the end values used in a problem may be chosen in different ways (see Ex. 3, Note 1); (3) the calculus method of computing areas should not be employed in a rule of thumb way, but with understanding and discretion (see Exs. 12, 13, 14).

Note 5. Precautions to be taken in finding areas and computing integrals. Suppose that the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates at A and B for which $x = a$ and $x = b$ respectively, is required. If the curve has an infinite ordinate between A and B, or if the ordinate is infinite at A or B, or at both A and B, or if either or both the end values a and b are infinite, the area may be finite or it may be infinite. It all depends on the curve; in one curve the area may be finite, in another curve it may be infinite. When infinite ordinates occur, either within or bounding the area whose measure is required, and also when the end-values are infinite, special care is necessary in applying the calculus to compute the area. The calculus method for finding areas and evaluating definite integrals can be used immediately with full confidence, only when the end values a and b are finite and when there is no infinite ordinate for any value of x from a to b inclusive. For illustrations showing the necessity for caution and special investigation in other cases see Murray's *Integral Calculus*, Art. 28, Exs. 3, 4, 5, 6, Art. 29; Gibson, *Calculus*, § 126; Snyder and Hutchinson, *Calculus*, Arts. 152, 155.

NOTE 6. For the determination of the areas of curves whose equations are given in polar coördinates, see Art. 136. The beginner is able to proceed to Art. 136 now.

EXAMPLES.

16. Calculate the actual increases in area described in the Note and in Exs. 2, 4, Art. 67.

17. Find the areas of the figures which have the following boundaries: (1) The curve $y = x^3$ and the line $4y = x$. (2) The parabola $y^2 + 8x$ and the line $x + y = 0$. (3) The semi-cubical parabola $y^2 = x^3$ and the line $y = 2x$. (4) The curves $y^2 = x^3$ and $x^2 = 4y$. (5) The axes and the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$. (6) The curve $x^2 + 6y = 0$ and the line $y + 3 = 0$. (7) The curve $(y+4)^2 + (x+3)^2 = 0$ and the line $x + 6 = 0$. (8) The hyperbola $xy = 1$ and the ordinates: (a) at $x = 1$, $x = 7$; (b) at $x = 1$, $x = 15$; (c) at $x = 1$ and $x = n$. (d) The hyperbola $xy = k^2$ and the ordinates at $x = a$ and $x = b$. (And the x-axis in each case.)

18. Find the area of the loop of the curve $8y^2 = x^4(3 + x)$.

19. Show that the area of the figure bounded by an arc of a parabola and its chord is two-thirds the area of a parallelogram, two of whose opposite sides are the chord and a segment of a tangent to the parabola.

[SUGGESTION: First take a parallelogram whose other sides are parallel to the axis of the parabola.]

Ex. 20. Prove that the area of a closed curve is represented by

$$\tfrac{1}{2}\int\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt \text{ [or } \tfrac{1}{2}\int(x\,dy - y\,dx)]$$

taken round the curve. (See Williamson, *Integral Calculus*, Art. 139; Gibson, *Calculus*, § 128.)

112. Volumes of solids of revolution. Suppose that the arc PQ of the curve

$$y = f(x),$$

revolves about the x-axis. It is required to find the volume enclosed by the surface generated by PQ in its revolution and the circular ends generated by the ordinates AP and BQ. (This is put briefly: *the volume generated by PQ.*) Let $OA = a$ and $OB = b$.

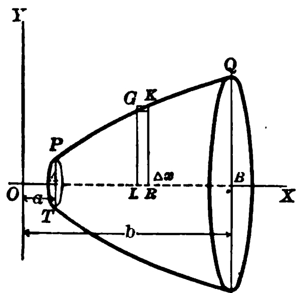

FIG. 55.

Suppose that AB is divided into any number of parts, say n, each equal to Δx. On any one of these parts, say LR, construct an "inner" and an "outer" rectangle, as shown in Fig. 55. Let G be the point (x, y), and K be the point $(x + \Delta x,\ y + \Delta y)$. When PQ revolves about the x-axis, the inner rectangle GR describes a cylinder of radius GL (*i.e.* y), and thickness Δx. At the same time the outer rectangle KL describes a cylinder of radius KR (*i.e.* $y + \Delta y$), and thickness Δx. It is evident that the volume $PQST$ is greater than the sum of the cylinders described by the inner rectangles, and is less than the sum of the cylinders described by the outer rectangles. That is,

sum of outer cylinders $>$ vol. $PQST >$ sum of inner cylinders.

The difference between the volume of the outer cylinders and the volume of the inner cylinders approaches zero when Δx approaches zero. Hence,

vol. $PQST = \lim_{\Delta x \doteq 0}$ {sum of inner (or outer) cylinders}.

That is,

vol. $\boldsymbol{PQST} = \lim_{\Delta x \doteq 0}$ {sum of cylinders like that generated by GR when x increases from a to b}

$$= \lim_{\Delta x \doteq 0} \sum_{x=a}^{x=b} (\pi \overline{LG}^2 \cdot \Delta x) = \pi \int_{x=a}^{x=b} y^2\, dx.$$

(See Art. 96.)

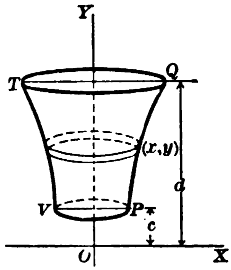

FIG. 56.

The infinitesimal differential $\pi y^2\,dx$, which is the volume of an infinitesimal cylinder of radius y and infinitesimal thickness dx, is called *an element of volume.*

When PQ revolves about the y-axis the element of volume is evidently $\pi x^2 dy$. If the ordinates of P and Q are c and d respectively, the volume generated,

$$\text{vol. } PQTV = \pi\int_{y=c}^{y=d} x^2\,dy.$$

NOTE 1. It is almost self-evident that the volume of the inner cylinders and the volume of the outer cylinders (Fig. 55), approach equality when their thickness Δx approaches zero.

NOTE 2. See Art. 67 (e).

EXAMPLES.

1. Find the volume generated by the revolution, about the x-axis, of the part of the line $3x + 10y = 30$ intercepted between the axes.

The given line is AB. The element of volume is $\pi y^2\,dx$. At B, $x = 0$; at A, $x = 10$. Accordingly, the end-values of x are 0 and 10. Hence,

$$\text{vol. cone } ABC = \pi\int_{x=0}^{x=10} y^2\,dx = \pi\int_0^{10}\left(\frac{30-3x}{10}\right)^2 dx$$
$$= 94.248 \text{ cubic inches.}$$

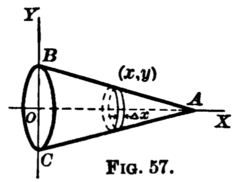

FIG. 57.

2. Verify the result in Ex. 1 by finding the volume of the cone in the ordinary way.

3. Derive by the calculus the ordinary formula for finding the volume of a right circular cone having height h and base of radius a. (See Ex. 8.)

4. (a) Find the volume generated by the revolution of the ellipse $9x^2 + 16y^2 = 144$ about the x-axis. (b) Find the volume bounded by a zone of the surface and the planes for which $x = 2$ and $x = 3$.

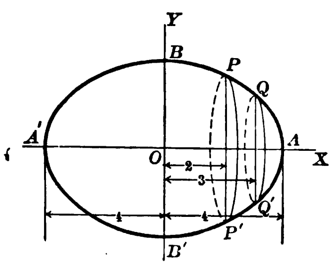

FIG. 58.

The element of volume is $\pi y^2\,dx$.

(a) Vol. ellipsoid

$$= 2 \text{ vol. } ABB' = 2\pi\int_{x=0}^{x=4} y^2\,dx$$
$$= \frac{2\pi}{16}\int_0^4 (144 - 9x^2)dx = 48\pi$$
$$= 150.8 \text{ cubic units.}$$

Or, $\quad$ vol. ellipsoid $= \pi \int_{x=-4}^{x=4} y^2\,dx \quad = 150.8$ cubic units.

(b) Vol. segment $PQQ'P' = \pi \int_{x=2}^{x=3} y^2\,dx = \frac{87}{16}\pi = 17.08$ cubic units.

5. Find the volume generated by revolving the arc of the curve $y = x^3$ between the points (0, 0) and (2, 8), about the y-axis.

The arc is OA. The element of volume, taking any point $P(x, y)$ on OA, is $\pi x^2\,dy$. Hence,

$$\text{vol. } OAB = \pi \int_{y=0}^{y=8} x^2\,dy = \pi \int_0^8 y^{\frac{2}{3}}\,dy = \tfrac{96}{5}\pi$$

$$= 60.32 \text{ cubic units.}$$

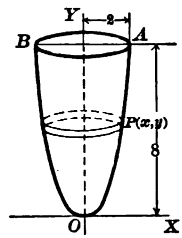

FIG. 59.

The integral may also be expressed in terms of x. Thus,

$$\text{vol. } OAB = \pi \int_{x=0}^{x=2} x^2\,dy.$$

Since $\quad y = x^3,\ dy = 3x^2\,dx.$

$$\therefore \text{ vol. } OAB = 3\pi \int_0^2 x^4\,dx = \tfrac{96}{5}\pi = 60.32, \text{ as above.}$$

6. Find the volume generated by revolving about the y-axis the arc of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

between the lines $x = a$ and $x = -a$. ACA' is the catenary; A and A' are the points whose abscissas are a and $-a$ respectively. *The volume generated by revolving ACA' about OY is evidently the same as the volume generated by revolving CA.* The element of volume is $\pi x^2\,dy$.

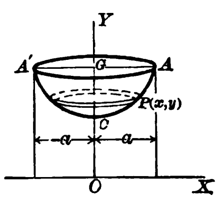

FIG. 60.

$$\therefore \text{ vol. } ACA'G = \pi \int_{x=0}^{x=a} x^2\,dy. \tag{1}$$

In this case it is easier to express the differential and the end-values in terms of x than in terms of y. From the equation of the curve it follows that

$$dy = \tfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)dx.$$

Hence (1) becomes vol. $ACA'G = \dfrac{\pi}{2}\displaystyle\int_0^a \left(x^2 e^{\frac{x}{a}} - x^2 e^{-\frac{x}{a}}\right)dx. \qquad (2)$

Integration (by parts) of the terms in (2) gives

$$\text{vol. } ACA'G = \frac{\pi a^3}{2}\left(e + \frac{5}{e} - 4\right) = .878\,a^3.$$

7. Find, by the calculus, the volume of the ring generated by revolving a circle of radius 5 inches about a line distant 7 inches from the centre of the circle.

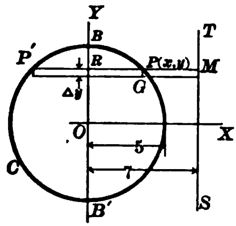

FIG. 61.

Let C be the circle and ST the line. Choose for the x-axis the line passing through the centre O at right angles to ST, and take OY for the y-axis. Then

the equation of the circle is $\qquad x^2 + y^2 = 25,$

and the equation of the line is $\qquad x = 7.$

Through any point $P(x, y)$ on the circle, draw $P'PM$ parallel to the x-axis. Suppose that PG, at right angles to PP', is of infinitesimal length dy. Then the rectangle $P'G$, on revolving about ST, generates an infinitesimal part of the volume of the ring. The limit of the sum of these parts as y changes from B' to B, is the volume required.

The volume generated by $P'G = \pi(\overline{P'M}^2 - \overline{PM}^2)\,dy.$

Now $$PM = 7 - PR = 7 - \sqrt{25 - y^2},$$

and $$P'M = 7 + RP' = 7 + \sqrt{25 - y^2}.$$

$\therefore$ vol. generated by $P'G = 28\,\pi\sqrt{25 - y^2}\cdot dy.$

$$\therefore \qquad \text{vol. of ring} = 2\int_{y=0}^{y=5} 28\,\pi\sqrt{25 - y^2}\,dy = 350\,\pi^2 \text{ cubic units.}$$

[Or, $$\text{vol. of ring} = \int_{y=-5}^{y=5} 28\,\pi\sqrt{25 - y^2}\,dy = 350\,\pi^2 \text{ cubic units,}$$
as in Ex. 4 (*a*).]

8. Find the volume of a cone in which the base is any plane figure of area A, and the perpendicular from the vertex to the base is h.

9. Find the volume generated by revolving the arc BEC (Fig. 53) about the x-axis.

10. Find the volume generated by the revolution of $MTKN$ (Fig. 52) about the x-axis.

11. Find the volume generated by the revolution of $ORLM$ (Fig. 51) about the y-axis.

12. Find the volume generated by the revolution of $ARLB$ (Fig. 50): (*a*) about the y-axis; (*b*) about the x-axis.

13. Find the volume generated by revolving the loop in Ex. 18, Art. 111, about the x-axis.

14. Find, by the calculus, the volume generated by the revolution about the x-axis, of the part of each of the following lines that is intercepted between the axes, and verify the results by the ordinary rule for finding the volume of a cone:

(1) $3x + 4y = 2$; (3) $7x + 3y + 20 = 0$;

(2) $2x - 5y = 7$; (4) $3x - 4y + 10 = 0$.

15. Find the volume generated by the revolution about the y-axis, of each of the intercepts in Ex. 14, and verify the result by the usual method of computation.

16. Find the volume generated when each of the figures described in Ex. 17, (1)–(9), Art. 111, revolves about the x-axis.

17. Find the volume generated when each of the figures in Ex. 16 revolves about the y-axis.

18. The figures bounded by a quadrant of an ellipse of semi-axes 9 and 5 inches and the tangents at its extremities revolves about each tangent in turn: find the volumes of each of the solids thus generated.

19. Find the volume of a sphere of radius a, considering the sphere as generated by the revolution of a circle about one of its diameters.

NOTE 3. The volume of a sphere may also be obtained by considering the sphere as made up of concentric spherical shells of infinitesimal thickness. The volume of a shell whose inner radius is r and whose thickness is an infinitesimal dr is (to within an infinitesimal of lower order) $4\pi r^2\,dr$. Accordingly, volume of sphere $= \int_0^a 4\pi r^2\,dr = \frac{4}{3}\pi a^3$.

20. Find the volume generated by the revolution of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis. (*Ans.* $\frac{32}{105}\pi a^3$.)

113. Derivation of the equations of curves. The equation of a curve or family of curves can be found when a geometrical property of a curve is known. Exercises of this kind constitute an important part of analytic geometry. For instance, the equation of a circle can be derived from the property that the points on the circle are at a given common distance from a fixed point. The statement of a geometrical property possessed by a curve may involve derivatives or differentials. To derive the equation of the curve from this statement is, quite frequently, a difficult problem. There are a few simple cases, however, in which it is possible to find the equation of the curve by means of a knowledge of the preceding articles. A few very simple examples have been given in Art. 97.

NOTE 1. It may be worth while merely to glance at more difficult problems of this kind and at the text relating thereto, in Chapter XXI. and in Murray's *Introductory Course in Differential Equations*, Chaps. V. and X. Also see Cajori, *History of Mathematics*, pp. 207–208, "Much greater than . . . integral of it."

NOTE 2. It has been shown in Arts. 58, 59, that for the curve whose equation is $f(x, y) = 0$, rectangular coördinates, if (x, y) denotes any point on the curve and m is the slope of the tangent at (x, y), then

$$m = \frac{dy}{dx};\quad \text{subtangent} = y\frac{dx}{dy};\quad \text{subnormal} = y\frac{dy}{dx}.$$

NOTE 3. It has been shown in Arts. 60, 61, that for the curve whose equation is $f(r, \theta) = 0$, if (r, θ) denotes any point on the curve, ψ the angle between the radius vector and the tangent at this point, and ϕ the angle which the tangent makes with the initial line, then

$$\tan\psi = r\frac{d\theta}{dr};\quad \phi = \psi + \theta;$$

$$\text{polar subtangent} = r^2\frac{d\theta}{dr};\quad \text{polar subnormal} = \frac{dr}{d\theta}.$$

N.B. *Draw the curves in the following examples.*

EXAMPLES.

1. A curve has a constant subnormal 4 and passes through the point (3, 5): what is its equation?

Here the subnormal, $\quad y\dfrac{dy}{dx} = 4.$

On using differentials, $\quad y\,dy = 4\,dx.$

Integration gives $\quad \dfrac{y^2}{2} + c_1 = 4x + c_2;$

whence $\quad \dfrac{y^2}{2} = 4x + k,$ in which $k = c_2 - c_1.$

Since (3, 5) is on the curve, $\frac{25}{2} = 12 + k$, whence $k = \frac{1}{2}$.

Accordingly, $\quad \dfrac{y^2}{2} = 4x + \dfrac{1}{2},$ *i.e.* $y^2 = 8x + 1$, is the equation.

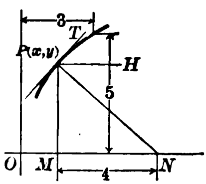

FIG. 62.

NOTE 4. In working these examples it is enlivening and helpful, to express the given conditions by means of a figure. This tentative figure can be corrected when fuller information is derived. Thus, for Ex. 1 draw a curve passing through (3, 5), and at any point $P(x, y)$ on this curve make the construction in Fig. 62 showing the subnormal 4. Here $\angle MPN = \angle HPT$. Now $\tan MPN = \dfrac{4}{y}$, *i.e.* $\dfrac{dy}{dx} = \dfrac{4}{y}$. Then proceed as above.

2. A curve has a constant subnormal and passes through the points $(2, 4)$, $(3, 8)$: find its equation and the length of the constant subnormal.

3. A curve has a constant subtangent 2, and passes through the point $(4, 1)$: find its equation.

4. Determine the curve which has a constant subtangent and passes through the points $(4, 1)$, $(8, e)$: find its equation and the length of the subtangent.

5. Find the curve in which the length of the subtangent for any point is twice the length of the abscissa, and which passes through $(3, 4)$.

6. In what curves does the subnormal vary as the abscissa? Determine the curve in which the length of the subnormal for any point is proportional to the length of the abscissa, and which passes through the points $(2, 4)$, $(3, 8)$.

7. In what curves does the slope vary as the abscissa? Determine the curve in which the slope at any point is proportional to the length of the abscissa, and which passes through the points $(0, 2)$, $(3, 5)$.

8. In what curves does the slope vary inversely as the ordinate? Determine the curve in which the slope at any point is inversely proportional to the length of the ordinate and which passes through the points named in Ex. 7.

9. Determine the polar curves in which the tangent at any point makes with the initial line an angle equal to twice the vectorial angle. Which of these curves passes through the point $\left(4, \frac{\pi}{2}\right)$?

10. Determine the polar curves in which the subtangent is twice the radius vector. Which of these curves passes through the point $(2, 0^c)$?

11. Determine the polar curves in which the subnormal varies as the sine of the vectorial angle, and which pass through the pole.

CHAPTER XIII.

INTEGRATION OF IRRATIONAL AND TRIGONOMETRIC FUNCTIONS.

114. The integration of differential expressions involving irrational quantities and trigonometric quantities will now be considered. Examples of this kind and methods of treating them have already been given in preceding articles. (See Art. 104, Art. 105, Exs. 10–18.) Only a few very special forms are discussed in this book.

NOTE. Chapter XI. provides a good part of the knowledge of formal integration sufficient for elementary work in physics and mechanics and for the ordinary problems in engineering. Accordingly, this chapter may be merely glanced at by those who have only a very short time to give to the study of the calculus and thus find it necessary to take on faith the results given in tables of integrals.

INTEGRATION OF IRRATIONAL FUNCTIONS.

115. The reciprocal substitution. This substitution, which sometimes leads to an easily integrable form, has been shown in Art. 107, Ex 6. Additional exercises are here appended.

Ex. 1. Find $\int \frac{dx}{x^2\sqrt{x^2-a^2}}$.

Put $x = \frac{1}{t}$. Then $dx = -\frac{1}{t^2}\,dt$; and

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = -\int \frac{t\,dt}{\sqrt{1-a^2t^2}} = \frac{1}{2\,a^2}\int (1-a^2t^2)^{-\frac{1}{2}}d(1-a^2t^2)$$

$$= \frac{1}{a^2}(1-a^2t^2)^{\frac{1}{2}} = \frac{\sqrt{x^2-a^2}}{a^2x}.$$

Exs. **2–9.** Derive integrals 23, 26, 27, 39, 42, 43, 54 *a*, 59 *a*, 61 *a*, pages 403–406.

NOTE. *Trigonometric substitutions.* Examples of a useful trigonometric substitution have been given in Art. 105, Exs. 4, 5. A differential expression in which $\sqrt{a^2+x^2}$ occurs may sometimes be simplified for purposes of integration by substituting $a\tan\theta$ for x, and expressions containing $\sqrt{x^2-a^2}$ by substituting $a\sec\theta$ for x.

For instance, in Ex. 1 put $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta\,d\theta$; and $\int\frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{1}{a^2}\sin\theta = \frac{\sqrt{x^2-a^2}}{a^2x}$.

116. Differential expressions involving $\sqrt[n]{a+bx}$. By this is meant differentials in which the irrational terms or factors are fractional powers of a single form, $a+bx$. (In particular cases a may be 0 and b may be 1; the irrational terms or factors are then fractional powers of x.) For preceding instances see Art. 105, Ex. 3, and Exs. 4, 10 at the end of Chapter XI.

If n is the least common denominator of the fractional indices of $a+bx$, the expression reduces to the form

$$F(x, \sqrt[n]{a+bx})\,dx. \tag{1}$$

This can be rationalised by putting

$$a+bx=z^n.$$

For then $x=\frac{z^n-a}{b}$ and $dx=\frac{n}{b}z^{n-1}dz$; and, accordingly, expression (1) becomes

$$\frac{n}{b}F\left(\frac{z^n-a}{b}, z\right)z^{n-1}\,dz.$$

This is rational in z, and accordingly may be integrated by the preceding articles.

Ex. 1. $\int\frac{x^{\frac{1}{2}}\,dx}{1+x^{\frac{3}{4}}}$.

Ex. 2. $\int\frac{\sqrt{x}\,dx}{x+1}$.

Ex. 3. $\int\frac{x\,dx}{\sqrt[3]{(3x-2)^4}}$.

Ex. 4. $\int(3+x)\sqrt{(2+x)^3}\,dx$.

Ex. 5. $\int\frac{dx}{\sqrt{2-x}(7+5\sqrt{2-x})}$.

Ex. 6. $\int\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\,dx$.

117 *A*. Expressions of the form $F(x, \sqrt{x^2+ax+b})\,dx$. *B*. Expressions of the form $F(x, \sqrt{-x^2+ax+b})\,dx$; $F(u, v)$ being a rational integral function of u and v.

A. The first expression can be rationalised by putting

$$\sqrt{x^2+ax+b} = z - x, \tag{1}$$

and changing the variable from x to z.

For, on squaring and solving Equation (1) for x,

$$x = \frac{z^2-b}{a+2z}. \tag{2}$$

From this,

$$dx = \frac{2(z^2+az+b)}{(a+2z)^2}\,dz. \tag{3}$$

On substituting the value of x in (2) in the second member of (1),

$$\sqrt{x^2+ax+b} = \frac{z^2+az+b}{a+2z}.$$

Accordingly,

$$F(x, \sqrt{x^2+ax+b})\,dx \text{ becomes } 2F\left(\frac{z^2-b}{a+2z}, \frac{z^2+az+b}{a+2z}\right)\frac{z^2+az+b}{(a+2z)^2}\,dz.$$

This is rational in z, and, accordingly, may be integrated by preceding articles.

Ex. 1. Find $\displaystyle\int\frac{x\,dx}{\sqrt{x^2-x+1}}$.

Assume $\qquad \sqrt{x^2-x+1} = z - x.$

From this, $\qquad x = \dfrac{z^2-1}{2z-1}.$

Then $\qquad dx = \dfrac{2(z^2-z+1)}{(2z-1)^2}\,dz,$

and $\qquad \sqrt{x^2-x+1} = z - x = \dfrac{z^2-z+1}{2z-1}.$

On substitution of these values in the given integral,

$$\int \frac{x\,dx}{\sqrt{x^2-x+1}} = 2\int \frac{z^2-1}{(2z-1)^3}dz = \tfrac{1}{4}z + \frac{3}{4(2z-1)} + \log\sqrt{2z-1} + c$$

(See Art. 108.)

$$= \frac{x+\sqrt{x^2-x+1}}{2} + \frac{3}{4(2x-1+2\sqrt{x^2-x+1})}$$

$$+ \tfrac{1}{2}\log(2x-1+2\sqrt{x^2-x+1}) + c$$

$$= \tfrac{1}{2}\log(2x-1+2\sqrt{x^2-x+1}) + \sqrt{x^2-x+1} + k.$$

$(k = \tfrac{1}{4} + c.)$

It happens that this is not the shortest way of working this particular example; but the above serves to show the substitution described in this article. The integral may also be obtained in the following way; this method is applicable to many integrals.

$$\int \frac{x\,dx}{\sqrt{x^2-x+1}} = \int \left(\frac{1}{2}\cdot\frac{2x-1}{\sqrt{x^2-x+1}} + \frac{1}{2}\cdot\frac{1}{\sqrt{x^2-x+1}}\right)dx$$

$$= \int \tfrac{1}{2}(x^2-x+1)^{-\frac{1}{2}}d(x^2-x+1) + \tfrac{1}{2}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}}$$

$$= \sqrt{x^2-x+1} + \tfrac{1}{2}\log(x-\tfrac{1}{2}+\sqrt{x^2-x+1}) + c$$

$$= \sqrt{x^2-x+1} + \tfrac{1}{2}\log(2x-1+2\sqrt{x^2-x+1}) + c_1.$$

Ex. 2. $\displaystyle\int \frac{(x-5)dx}{\sqrt{x^2-6x+25}} = \int\left(\frac{x-3}{\sqrt{x^2-6x+25}} - \frac{2}{\sqrt{(x-3)^2+16}}\right)dx$

$$= \sqrt{x^2-6x+25} - 2\log(x-3+\sqrt{x^2-6x+25}).$$

B. Suppose that $\quad -x^2+ax+b = (x-p)(q-x).$
The second expression at the head of this article can be rationalised by putting

$$\sqrt{-x^2+ax+b},\ \textit{i.e.}\ \sqrt{(x-p)(q-x)} = (x-p)z, \tag{3}$$

and changing the variable from x to z.

On squaring in (3), $\quad q - x = (x-p)z^2;$

on solving for x, $$x = \frac{pz^2+q}{1+z^2}; \tag{4}$$

whence, on differentiation, $\quad dx = \dfrac{2z(p-q)}{(1+z^2)^2}dz.$

Substitution of the value of x in (4) in the second member of (3), gives

$$\sqrt{-x^2+ax+b}=\frac{(q-p)z}{1+z^2}.$$

Accordingly,

$$F(x, \sqrt{-x^2+ax+b})dx \text{ becomes } 2(p-q)F\left(\frac{pz^2+q}{1+z^2}, \frac{(q-p)z}{1+z^2}\right)\frac{z\,dz}{(1+z^2)^2}.$$

This is rational in z, and, accordingly, may be integrated by preceding articles.

NOTE 1. Instead of (3) the relation

$$\sqrt{(x-p)(q-x)}=(q-x)z$$

may be used.

NOTE 2. If $\sqrt{\pm px^2+qx+r}$ occurs, it may be reduced to form A or B; thus, $\sqrt{p}\sqrt{\pm x^2+\frac{q}{p}x+\frac{r}{p}}$.

EXAMPLES.

3. Find $\int\frac{dx}{x\sqrt{12-x-x^2}}$.

Put $\sqrt{12-x-x^2}=\sqrt{(x+4)(3-x)}=(x+4)z$.

From this, on squaring, $3-x=(x+4)z^2$.

On solving for x, $x=\frac{3-4z^2}{1+z^2}$.

Accordingly, $dx=\frac{-14z\,dz}{(1+z^2)^2}$, $\sqrt{12-x-x^2}=(x+4)z=\frac{7z}{1+z^2}$.

$$\therefore \int\frac{dx}{x\sqrt{12-x-x^2}}=2\int\frac{dz}{4z^2-3}=\frac{1}{2\sqrt{3}}\log\frac{2z-\sqrt{3}}{2z+\sqrt{3}}$$

$$=\frac{1}{2\sqrt{3}}\log\frac{2\sqrt{3-x}-\sqrt{3(x+4)}}{2\sqrt{3-x}+\sqrt{3(x+4)}}.$$

4. Solve Ex. 3, using the substitution $\sqrt{12-x-x^2}=(3-x)z$.

5. $\int\frac{(2x+5)\,dx}{\sqrt{4x^2+6x+11}}$.

6. $\int\frac{(3x-4)\,dx}{\sqrt{12-4x-x^2}}$.

7. $\int\frac{dx}{x\sqrt{12-4x-x^2}}$.

8. $\int\frac{(3x-4)\,dx}{x\sqrt{12-4x-x^2}}$. $\left[\frac{3x-4}{x}=3-\frac{4}{x}.\right]$

9. $\int \frac{dx}{x\sqrt{x^2+x+1}}$. 10. $\int \frac{(x^2+2x-3)\,dx}{x\sqrt{x^2+x+1}}$.

11. $\int \frac{dx}{(x+2)\sqrt{x^2+4x-12}}$. (Put $x+2=z$.)

NOTE 3. The integrands in integrals of the form $\int x^p(a+bx^2)^{\frac{2m+1}{2}}dx$ in which m is any integer and p is an *odd* integer, positive or negative, can be rationalised by means of the substitution $a+bx^2=z^2$. Thus:

12. $\int \frac{x^3\,dx}{\sqrt{x^2-a^2}}$.

Put $$x^2-a^2=z^2.$$

Then $$x\,dx=z\,dz;$$

and $$\int \frac{x^3\,dx}{\sqrt{x^2-a^2}}=\int (z^2+a^2)dz=\frac{z}{3}(z^2+3a^2)=\frac{x^2+2a^2}{3}\sqrt{x^2-a^2}.$$

13. Find $\int \frac{dx}{x\sqrt{x^2-a^2}}$ (see Formula XXI., Art. 107): (1) Using the substitution $x=a\sec\theta$; (2) using the substitution $x=\frac{1}{t}$; (3) using the substitution $x^2-a^2=z^2$. (Show the equivalence of the various forms of the integral.)

14. Show the truth of the statement in Note 3.

118. To find $\int x^m(a+bx^n)^p dx$. Here m, n, and p are *constants*, positive or negative, integral or fractional. The given integral, as will be shown in the working of examples, can be connected with simpler integrals in a particular way. By "a simpler integral" is meant one that is simpler from the point of view of integration. For instance, if $m=5$, the integral in which $m=3$, other things being the same, is simpler; if $p=-\frac{3}{2}$, the integral in which $p=-\frac{1}{2}$, other things being the same, is simpler. It will be found that *the given integral can be connected with an integral in which the m is increased or decreased by n, or with an integral in which the p is increased or decreased by* 1; *i.e.*, with one or other of the four integrals:

$$\left.\begin{array}{ll}\int x^{m+n}(a+bx^n)^p\,dx, & \int x^m(a+bx^n)^{p+1}\,dx,\\ \int x^{m-n}(a+bx^n)^p\,dx, & \int x^m(a+bx^n)^{p-1}\,dx.\end{array}\right\}\qquad(a)$$

When one of these four integrals is chosen, a relation between it and the required integral can be expressed in the following way:

Form a function of x in which the x outside the bracket has an index one greater than the least index of the corresponding x in the required and the chosen integrals, and in which the bracket has an index one greater than the least index of the bracket in those integrals. Give the function thus formed an arbitrary constant coefficient and give the chosen integral an arbitrary constant coefficient; equate the sum of these quantities to the required integral. The value of the arbitrary coefficients can then be determined.

For example, let

$$\int x^m(a+bx^n)^p dx \text{ be connected with } \int x^m(a+bx^n)^{p-1}dx.$$

The function formed by the rule is $x^{m+1}(a+bx^n)^p$. Put

$$\int x^m(a+bx^n)^p dx = Ax^{m+1}(a+bx^n)^p + B\int x^m(a+bx^n)^{p-1}dx, \quad (1)$$

in which A and B are arbitrary constants.

It is now necessary to find such values for A and B as will make (1) an identical equation.

In order to determine A and B, take the derivatives of both members of (1), simplify, and then equate coefficients of like powers of x. Thus, on differentiating the members of (1),

$$x^m(a+bx^n)^p = A(m+1)x^m(a+bx^n)^p + Ax^{m+1}p(a+bx^n)^{p-1}nbx^{n-1} \\ + Bx^m(a+bx^n)^{p-1}.$$

On division by $x^m(a+bx^n)^{p-1}$, and simplification,

$$a+bx^n = Ab(m+np+1)x^n + Aa(m+1) + B.$$

On equating coefficients of like powers of x and solving for A and B,

$$A = \frac{1}{m+np+1}, \qquad B = \frac{anp}{m+np+1}.$$

The substitution of these values in (1) gives

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p}{m+np+1} + \frac{anp}{m+np+1}\int x^m(a+bx^n)^{p-1}\,dx.$$

On connecting the required integral with each of the other integrals in (a) and proceeding in a similar manner, the results (1), (2), (4), page 401, are obtained. The deduction of them is left as an exercise for the student.

NOTE 1. Formulas 1–4, page 401, are examples of what are usually termed **Formulas of Reduction.** Frequently integrals are obtained by substituting the particular values of m, n, p in these formulas of reduction. To memorize such formulas is, however, a waste of energy; it is better, at least for beginners, to integrate *by the method* whereby these formulas have been obtained.

NOTE 2. It will be observed that some of these formulas fail for certain values of m, n, p; viz., when $m+np+1=0$, when $m=-1$, and when $p=-1$. Other formulas or other methods may be applied in each of these cases.

NOTE 3. Its success may be regarded as one proof of the above method. In the large majority of text-books on calculus, formulas 1, 2, 3, 4, page 401, are derived in a straightforward way by integration by parts. For this derivation see almost any calculus, *e.g.* Murray, *Integral Calculus*, Appendix, Note B. For other formulas of reduction for $\int x^m(a+bx^n)^p\,dx$, obtained by the method of "connection" or "arbitrary coefficients," see Edwards, *Integral Calculus*, Art. 82, and integrals 5, 6, page 402.

EXAMPLES.

1. Find $\int\frac{dx}{x^2\sqrt{x^2-a^2}}$, *i.e.* $\int x^{-2}(x^2-a^2)^{-\frac{1}{2}}dx$. (See Ex. 1, Art. 115.)

Here $m=-2$, $n=2$, $p=-\frac{1}{2}$. The best integral to connect with is obviously the integral in which the m is raised by 2, viz. $\int\frac{dx}{\sqrt{x^2-a^2}}$. On making the connection according to the directions given above,

(1) $$\int x^{-2}(x^2-a^2)^{-\frac{1}{2}}\,dx = Ax^{-1}(x^2-a^2)^{\frac{1}{2}} + B\int(x^2-a^2)^{-\frac{1}{2}}\,dx.$$

It is now necessary to find such values for A and B as will make this equation an identical equation.

On differentiation, and equating the derivatives,

$$x^{-2}(x^2-a^2)^{-\frac{1}{2}} = -Ax^{-2}(x^2-a^2)^{\frac{1}{2}} + A(x^2-a^2)^{-\frac{1}{2}} + B(x^2-a^2)^{-\frac{1}{2}}.$$

On simplifying, by multiplying through by $x^2(x^2-a^2)^{\frac{1}{2}}$,

$$1 = -A(x^2-a^2) + Ax^2 + Bx^2.$$

On equating coefficients of like powers of x,

$$B = 0 \text{ and } Aa^2 = 1\text{; whence } A = \frac{1}{a^2}.$$

On substitution of these values of A and B in (1),

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}.$$

2. Find $\int \frac{x^3\,dx}{\sqrt{x^2-a^2}}$, *i.e.* $\int x^3(x^2-a^2)^{-\frac{1}{2}}\,dx$. (See Ex. 12, Art. 117.)

Here $m = 3$, $n = 2$, $p = -\frac{1}{2}$. It will obviously be an advantage to lessen m. Accordingly, let connection be made with $\int x(x^2-a^2)^{-\frac{1}{2}}\,dx$. On doing this in the way described,

$$(1) \qquad \int x^3(x^2-a^2)^{-\frac{1}{2}}\,dx = Ax^2(x^2-a^2)^{\frac{1}{2}} + B\int x(x^2-a^2)^{-\frac{1}{2}}\,dx.$$

It is now necessary to find such values for A and B as will make this an identical equation.

On taking the derivatives and equating them,

$$x^3(x^2-a^2)^{-\frac{1}{2}} = 2\,Ax(x^2-a^2)^{\frac{1}{2}} + Ax^3(x^2-a^2)^{-\frac{1}{2}} + Bx(x^2-a^2)^{-\frac{1}{2}}.$$

On simplifying, by dividing through by $x(x^2-a^2)^{-\frac{1}{2}}$,

$$x^2 = 2\,A(x^2-a^2) + Ax^2 + B.$$

On equating coefficients of like powers of x, and solving for A and B, it is found that $A = \frac{1}{3}$, $B = \frac{2}{3}a^2$.

Substitution in (1) gives

$$\int \frac{x^3\,dx}{\sqrt{x^2-a^2}} = \tfrac{1}{3}x^2\sqrt{x^2-a^2} + \tfrac{2}{3}a^2\int \frac{x\,dx}{\sqrt{x^2-a^2}}$$

$$= \tfrac{1}{3}x^2\sqrt{x^2-a^2} + \tfrac{2}{3}a^2\sqrt{x^2-a^2} = \frac{(x^2+2\,a^2)\sqrt{x^2-a^2}}{3}.$$

3. Find $\int \frac{dx}{(x^2+a^2)^k}$, *i.e.* $\int (x^2+a^2)^{-k}\,dx$.

Here $m = 0$, $n = 2$, $p = -k$. In this case it is better to increase p. On proceeding according to the rule,

$$(1) \qquad \int (x^2+a^2)^{-k}\,dx = Ax(x^2+a^2)^{-k+1} + B\int (x^2+a^2)^{-k+1}\,dx.$$

On differentiation, simplification of the resulting equation by division by $(x^2 + a^2)^{-k}$, equating coefficients of like powers, solving for A and B, and substitution of their values in (1), it will be found that

$$\int \frac{dx}{(x^2+a^2)^k} = \frac{1}{2\,a^2(k-1)}\left\{\frac{x}{(x^2+a^2)^{k-1}} + (2\,k-3)\int\frac{dx}{(x^2+a^2)^{k-1}}\right\}.$$

4. Derive $\int \sqrt{a^2 - x^2}\,dx$ by this method. (See Ex. 5, Art. 105, Ex. 5, Art. 106.)

5. Do Ex. 16, Art. 107, by this method.

Note 4. It is sometimes necessary to repeat the operation of reduction two or more times.

6. Derive integrals 21, 22, 23, 28, 30, 35, 40, 41, 42, 44, pages 403–404, and others of the collection.

7. Derive integrals 48, 53, 54, 55, 57, pages 405–406 $[\sqrt{2\,ax \pm x^2} = x^{\frac{1}{2}}(2\,a \pm x)^{\frac{1}{2}}]$. (Compare Exs. 6, 7, and Exs. 2–9, Art. 115.)

8. **Derive formulas 1–6, page 401.**

9. Find $$\int \frac{\sqrt{a^2-x^2}}{x^4}\,dx = -\frac{(a^2-x^2)^{\frac{3}{2}}}{3\,a^2x^3};$$

$$\int \frac{x^4}{\sqrt{a^2-x^2}}\,dx = \frac{1}{8}\left\{3\,a^4\sin^{-1}\frac{x}{a} - x(2\,x^2+3\,a^2)\sqrt{a^2-x^2}\right\}.$$

10. Using integrals 1–4 as formulas of substitution for the values of m, n, p, a, b, derive some of the integrals 21–30, 37–46, 53–61, pages 403–406.

Note 5. On the integration of irrational expressions also see Snyder and Hutchinson, *Calculus*, Arts. 129–131, 139, 140. These articles convey valuable additional information, and, in particular, Art. 139 gives an interesting *geometrical interpretation* concerning the rationalisation of the square root of a quadratic expression. Also see the references given in Art. 122, Note 2.

INTEGRATION OF TRIGONOMETRIC FUNCTIONS.

N.B. On account of the numerous relations between the trigonometric ratios, the indefinite integral of a trigonometric differential can take many forms.

119. Algebraic transformations. A differential expression involving only trigonometric ratios can be transformed into an algebraic differential by substituting a variable, t say, for one of the trigonometric ratios. The algebraic differential thus obtained may possibly be integrated by some method shown in the preceding articles. Knowledge as to what substitution will be the most

convenient one to make in a given case can best be acquired by trial and experience. Illustrations of this article have already been met in Art. 105, Exs. 10, 11, 16, 17.

Ex. 1. See exercises just referred to.

Ex. 2. Do Exs. 1–5, 7–9, Art. 120, making algebraic transformations.

120. Integrals reducible to $\int F(u)\,du$, in which u is one of the trigonometric ratios.

(*a*) *$\int \sin^n x\,dx$ and $\int \cos^n x\,dx$ are thus reducible when n is an odd positive integer.* For

$$\int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx = -\int (1-\cos^2 x)^{\frac{n-1}{2}}\,d(\cos x).$$

The latter form can be expanded in a finite number of terms, $\frac{n-1}{2}$ being an integer, and then integrated term by term. $\int \cos^n x\,dx$ can be treated similarly.

EXAMPLES.

1. $\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx = \int (1-\sin^2 x)^2\,d(\sin x)$

$= \int (1 - 2\sin^2 x + \sin^4 x)\,d(\sin x) = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c.$

2. $\int \sin^3 x\,dx$, $\int \cos^3 x\,dx$, $\int \sin^5 x\,dx$.

(*b*) *$\int \sin^n x \cos^m x\,dx$ is thus reducible when either n or m is a positive odd integer.*

3. $\int \sin^3 x \cos^{\frac{5}{2}} x\,dx = \int \sin^2 x \cos^{\frac{5}{2}} x \sin x\,dx$

$= -\int (1-\cos^2 x)\cos^{\frac{5}{2}} x\,d(\cos x) = -\int (\cos^{\frac{5}{2}} x - \cos^{\frac{9}{2}} x)\,d(\cos x)$

$= -\frac{2}{7}\cos^{\frac{7}{2}} x + \frac{2}{11}\cos^{\frac{11}{2}} x + c.$

4. (1) $\int \frac{\sin^3 x}{\sqrt[3]{\cos x}}\,dx$, (2) $\int \cos^5 x \sin^{\frac{3}{5}} x\,dx$, (3) $\int \frac{\cos^5 x\,dx}{\sqrt{\sin x}}$,
(4) $\int \cos^{\frac{2}{3}} x \sin^3 x\,dx$.

NOTE. Case (*a*) is a special case of (*b*).

(*c*) $\int \sec^n x\,dx$ *and* $\int \operatorname{cosec}^n x\,dx$ *are thus reducible when n is a positive even integer.*

5. $\int \operatorname{cosec}^6 x\,dx = \int \operatorname{cosec}^4 x \cdot \operatorname{cosec}^2 x\,dx = -\int (1+\cot^2 x)^2\,d(\cot x)$

$$= -\cot x\,(1 + \tfrac{2}{3}\cot^2 x + \tfrac{1}{5}\cot^4 x).$$

6. Show the truth of statement (*c*).

7. (1) $\int \sec^4 x\,dx$, (2) $\int \operatorname{cosec}^4 x\,dx$, (3) $\int \sec^6 x\,dx$.

(*d*) $\int \tan^m x \sec^n x\,dx$ *and* $\int \cot^m x \operatorname{cosec}^n x\,dx$ *are thus reducible when n is a positive even integer, or when m is a positive odd integer.*

8. Show the truth of statement (*d*).

9. (1) $\int \tan^2 x \sec^4 x\,dx$, (2) $\int \sec^6 x\sqrt{\tan x}\,dx$, (3) $\int \tan^{\frac{5}{3}} x \sec^4 x\,dx$, (4) $\int \tan^5 x \sec^3 x\,dx$, (5) $\int \cot^3 x\sqrt{\operatorname{cosec} x}\,dx$, (6) $\cot^5 x \operatorname{cosec}^3 x\,dx$.

121. Integration aided by multiple angles. It is shown in trigonometry that

$$\sin u \cos u = \tfrac{1}{2}\sin 2u,$$
$$\sin^2 u = \tfrac{1}{2}(1 - \cos 2u),$$
$$\cos^2 u = \tfrac{1}{2}(1 + \cos 2u).$$

Accordingly, if n and m are positive even integers, $\sin^n x$, $\cos^n x$, and $\sin^n x \cos^m x$ can be transformed into expressions which are rational trigonometric functions of $2x$. Differential expressions involving the latter are, in general, more easily integrable than the original differential expressions in x.

Ex. 1. $\int \cos^4 x\,dx = \int \{\tfrac{1}{2}(1+\cos 2x)\}^2\,dx = \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx.$ Now $\int 2\cos 2x\,dx = \sin 2x$, and $\int \cos^2 2x\,dx = \tfrac{1}{2}\int (1+\cos 4x)\,dx = \tfrac{1}{2}(x + \tfrac{1}{4}\sin 4x)$. $\therefore \int \cos^4 x\,dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + c.$

Ex. 2. $\int \sin^2 x \cos^2 x\,dx = \tfrac{1}{4}\int \sin^2 2x\,dx = \tfrac{1}{8}\int (1 - \cos 4x)\,dx$

$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + c.$$

Ex. 3. (1) $\int \sin^4 x\,dx$, (2) $\int \cos^6 x\,dx$, (3) $\int \sin^4 x \cos^2 x\,dx$, (4) $\int \sin^3 x \cos^3 x\,dx$, (5) $\int \sin^4 x \cos^4 x\,dx$.

122. Reduction formulas. There are several formulas which are useful in integrating trigonometric differentials. A few of them are deduced here; the deduction of the others is left as an exercise for the student.

(*a*) *To find* $A: \int \sin^n x\,dx$, *and* $B: \int \cos^n x\,dx$, *when* n *is any integer.*

A. Integrate by parts, putting

$$dv = \sin x\,dx; \quad \text{then } u = \sin^{n-1} x,$$

$$v = -\cos x, \qquad du = (n-1)\sin^{n-2} x \cos x\,dx.$$

$$\therefore \int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.$$

From this, on transposition and division by n,

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx. \qquad (1)$$

This is a useful formula of reduction when n is a positive integer. From it can be deduced a formula which is useful when the index is a negative integer. For, on transposition and division by $\frac{n-1}{n}$, formula (1) becomes

$$\int \sin^{n-2} x\,dx = \frac{\sin^{n-1} x \cos x}{n-1} + \frac{n}{n-1}\int \sin^n x\,dx.$$

This result is true for all values of n, and, accordingly, for $n = N+2$. On putting $N+2$ for n, this becomes

$$\int \sin^N x\,dx = \frac{\sin^{N+1} x \cos x}{N+1} + \frac{N+2}{N+1}\int \sin^{N+2} x\,dx. \qquad (2)$$

If N is a negative integer, say $-m$, (2) may be written

$$\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\,\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}. \qquad (3)$$

In the above way calculate the following integrals:

Ex. 1. (1) $\int \sin^2 x\,dx$, (2) $\int \sin^3 x\,dx$, (3) $\int \sin^4 x\,dx$, (4) $\int \sin^5 x\,dx$.

Ex. 2. (1) $\int \frac{dx}{\sin^2 x}$, (2) $\int \frac{dx}{\sin^3 x}$, (3) $\int \frac{dx}{\sin^4 x}$.

Ex. 3. Compare the results in Exs. 1, 2, with those obtained for these integrals by methods of the preceding articles.

B. Similarly to *A* there can be deduced results 69, 71, page 407, for *B*. Formula 69 is useful for positive indices, and 71 for negative indices.

Ex. 4. Deduce formulas 69 and 71.

Ex. 5. (1) $\int \cos^4 x\,dx$, (2) $\int \cos^5 x\,dx$, (3) $\int \frac{dx}{\cos^4 x}$, (4) $\int \frac{dx}{\cos^5 x}$.

Compare results with those obtained by methods of preceding articles.

(*b*) *To find* $\int \sec^n x\,dx$ *when* n *is a positive integer greater than* 1.

Put $\sec^2 x\,dx = dv$; then $\sec^{n-2} x = u$,

$$\tan x = v, \quad (n-2)\sec^{n-2} x \tan x\,dx = du.$$

$$\therefore \int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx.$$

From this, on substituting $\sec^2 x - 1$ for $\tan^2 x$, and solving for $\int \sec^n x\,dx$,

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

Similarly, result 73 for $\int \operatorname{cosec}^n x\,dx$ can be obtained.

Ex. 6. (1) $\int \sec^3 x\,dx$, (2) $\int \sec^4 x\,dx$, (3) $\sec^5 x\,dx$.

Ex. 7. (1) $\int \csc^3 x\,dx$, (2) $\int \csc^4 x\,dx$, (3) $\csc^5 x\,dx$.

Ex. 8. Derive formula 73.

Ex. 9. From formulas 72 and 73 derive formulas for $\int \sec^n x\,dx$ and $\int \operatorname{cosec}^n x\,dx$ which are applicable when n is a negative integer.

[SUGGESTION: Use method employed in deducing formulas 70 and 71.]

(c) *To find* $\int \tan^n x\,dx$, *in which* n *is a positive integer greater than* 1.

$$\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x\,d(\tan x) - \int \tan^{n-2} x\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$$

Similarly can be shown result 75 for $\int \cot^n x\,dx$.

When n is negative, say $-m$, then $\int \tan^n x\,dx = \int \cot^m x\,dx$, and $\int \tan^n x\,dx$ can be expressed in cotangents by formula 75. Formulas applicable to cases in which n is negative, can be deduced from formulas 74 and 75, by the method used in deducing formulas 70 and 71.

Ex. 10. Deduce Formula 75, and formulas applicable to $\int \tan^n x\,dx$ and $\int \cot^n x\,dx$ when n is negative.

Ex. 11. (1) $\int \tan^3 x\,dx$, (2) $\cot^4 x\,dx$, (3) $\int \tan^4 x\,dx$, (4) $\int \tan^5 x\,dx$.

(*d*) $\int \sin^m x \cos^n x\,dx$. When m and n are integers, *reduction formulas* can be derived for this integral in a manner similar to that used in Art. 118; that is, by

(i) Connecting it with each of the four integrals in turn, viz.:

$$\int \sin^{m-2} x \cos^n x\,dx, \qquad \int \sin^m x \cos^{n-2} x\,dx,$$

$$\int \sin^{m+2} x \cos^n x\,dx, \qquad \int \sin^m x \cos^{n+2} x\,dx;$$

(ii) Forming a new function by giving $\sin x$ and $\cos x$ each an index one greater than the lesser of its indices in the required integral and the integral with which it is connected, and taking the product;

(iii) Giving the connected integral and this newly formed function each an arbitrary coefficient, and equating their sum to the required integral;

(iv) Determining the value of these coefficients by proceeding as in Art. 118.

The derivation of these reduction formulas is left as an exercise for the student; they are given in the set of integrals, **Nos. 76–79.***

Ex. 12. Deduce formulas Nos. 76–79 by the methods outlined above.

Ex. 13. Deduce the formulas in Ex. 12 by integrating by parts.

Ex. 14. Apply these formulas to finding the following integrals:

$$(1) \int \sin^2 x \cos^2 x\, dx; \quad (2) \int \cos^4 x \sin^2 x; \quad (3) \int \frac{\cos^4 x}{\sin^2 x}\, dx.$$

Ex. 15. Deduce the integrals in Ex. 14 by the method outlined in (*d*).

NOTE 1. When $m + n$ is a negative even integer, the above integral can be expressed in the form $\int f(\tan x)d(\tan x)$.

Ex. 16. $\displaystyle\int \frac{\sin^3 x}{\cos^7 x}\, dx = \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^4 x} \cdot dx = \int \tan^3 x \sec^4 x\, dx$

$$= \int \tan^3 x (1 + \tan^2 x) d \tan x = \tfrac{1}{24}(6 + 4 \tan^2 x) \tan^4 x.$$

Ex. 17. $\displaystyle (1) \int \frac{\cos^4 x}{\sin^8 x}\, dx, \quad (2) \int \frac{\sin^3 x}{\cos^5 x}\, dx, \quad (3) \int \frac{\cos^2 x}{\sin^6 x}\, dx.$

NOTE 2. **Special forms.** Integrals 80–87 are occasionally required. For their deduction see Murray, *Integral Calculus*, Arts. 54–57, or other texts. It will be a good exercise for the student to try to deduce these integrals himself. For a fuller discussion of the integration of irrational and trigonometric functions see the article *Infinitesimal Calculus* (Ency. Brit., 9th edition), §§ 124 on; also see Echols, *Calculus*, Chap. XVIII.

NOTE 3. *On* **integration by infinite series.** See Art. 126.

NOTE 4. **Elliptic integrals. Elliptic functions.** The algebraic integrands considered in this book give rise only to the ordinary *algebraic*, *circular*, and *hyperbolic*† functions. (The two last named are *singly periodic* functions.) Certain irrational integrands give rise to a class of functions treated in higher mathematics, viz. the *elliptic* (or *doubly periodic*) functions. The term *elliptic functions* is somewhat of a misnomer; for the elliptic functions are not connected with an ellipse in the same way as the circular functions are connected with the circle, and the hyperbolic functions with the hyperbola. The *elliptic integrals* derived their name from the fact that an integral of this kind appeared in the determination of the length of an arc of the ellipse. Out of the study of the elliptic integrals arose the modern

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* These formulas are derived in Murray, *Integral Calculus*, Art. 51, and Appendix, Note C. Also see Edwards, *Integral Calculus*, Art. 83.

† See Appendix, Note A.

extensive and important subject of *elliptic functions;* this accounts for the term *elliptic* in the name of these functions. The student may take a glance forward and extend his mathematical outlook by inspecting Art. 174, Note 4; Cajori, *History of Mathematics*, pages 279, 347–354; the section on elliptic integrals in the article mentioned in Note 2, in particular, §§ 191, 192, 204, 205, 206, 219, 220; W. B. Smith, *Infinitesimal Analysis*, Vol. I., Arts. 123–125; Glaisher, *Elliptic Functions*, pages 6, 175, etc.

EXAMPLES.

1. Derive integrals Nos. 80–82, 85–87.

2. Derive several of the integrals 18–30, 36–46, 53–65.

3. (1) $\int \frac{\sqrt[3]{t}}{t-1}\,dt.$ (2) $\int \frac{v\,dv}{(2v+1)^{\frac{3}{2}}}.$ (3) $\int \frac{dx}{(1+x^2)\sqrt{1-4x^2}}.$

(4) $\int \frac{x\,dx}{(1+x^2)\sqrt{1-4x^2}}.$ (5) $\frac{\sqrt{4x-x^2}}{x^2}\,dx.$ (6) $\int \frac{(2x+1)\,dx}{\sqrt{x^2+3x+5}}.$

(7) $\int \frac{(2x+1)\,dx}{x\sqrt{x^2+3x+5}}.$ (8) $\int \frac{dv}{(v+1)\sqrt{v^2+1}}.$ (9) $\int \frac{x\,dx}{(x^4-16)^3}.$

(10) $\int \frac{dx}{(x^2+4)^3}.$ (11) $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}.$ (12) $\int \frac{\sqrt{6x-x^2}}{x^2}\,dx.$

4. Derive the following integrals:

(1) $\int \sqrt{\frac{a+x}{a-x}}\,dx = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2}.$

(2) $\int \sqrt{\frac{a+x}{b-x}}\,dx = -\sqrt{(a+x)(b-x)} - (a+b)\sin^{-1}\sqrt{\frac{b-x}{a+b}}.$

(3) $\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)} + (a+b)\sin^{-1}\sqrt{\frac{x+b}{a+b}}.$

(4) $\int \sqrt{\frac{a+x}{b+x}}\,dx = \sqrt{(a+x)(b+x)} + (a-b)\log(\sqrt{a+x}+\sqrt{b+x}).$

(5) $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}}.$

5. Show that, if $f(u, v)$ is a rational function of u and v, and m and n are integers, then $f\{x^2, (a+bx^2)^{\frac{m}{n}}\}x\,dx$ can be rationalised by means of the substitution $a + bx^2 = z^n$. (Ex. 14, or Note 3, Art. 117, is a particular case of this theorem.)

6. Show that (1) $\int_0^{\frac{\pi}{2}} \sin^{2m} dx = \frac{1\cdot 3\cdot 5 \cdots (2m-1)}{2\cdot 4\cdot 6 \cdots 2m}\cdot\frac{\pi}{2};$

(2) $\int_0^{\frac{\pi}{2}} \sin^{2m+1}x\,dx = \frac{2\cdot 4\cdot 6\cdots 2m}{3\cdot 5\cdot 7\cdots(2m+1)}$ (m being an integer).

CHAPTER XIV.

APPROXIMATE INTEGRATION. MECHANICAL INTEGRATION.

123. Approximate integration of definite integrals. It has been shown in Arts. 95, 96, 98, that: (*a*) the definite integral $\int_a^b f(x)dx$ may be evaluated by finding the anti-differential of $f(x)dx$, $\phi(x)$ say, and calculating $\phi(b) - \phi(a)$; (*b*) this last number is also the measure of the area of the figure bounded by the curve $y = f(x)$, the x-axis, and the two ordinates for which $x = a$ and $x = b$. In only a few cases, however, can the anti-differential of $f(x)dx$ be found; in other cases an approximate value of the definite integral can be obtained by making use of fact (*b*). Thus, on the one hand the evaluation of a definite integral serves to give the measurement of an area; on the other hand the accurate measurement of a certain area will give the exact value of a definite integral, and an approximate determination of this area will give an approximate value of the integral. The area described above may be found approximately by one of several methods; two of these methods are explained in Arts. 124 and 125.

124. Trapezoidal rule for measuring areas (and evaluating definite integrals). Let the value of the definite integral $\int_a^b f(x)dx$ be required. Plot the curve $y = f(x)$ from $x = a$ to $x = b$. Let $OA = a$, $OB = b$, and draw the ordinates AP and BQ. By Art. 96, the measure of the area $APQB$ is the value of the required integral. An approximate value of the area $APQB$ can be found in the following

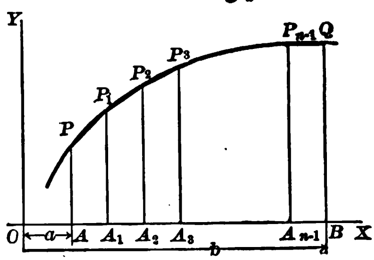

FIG. 63.

way. Divide the base AB into n intervals each equal to Δx, and at the points of division A_1, A_2, A_3, $\cdots$, erect ordinates A_1P_1, A_2P_2, A_3P_3, $\cdots$. Draw the chords PP_1, P_1P_2, P_2P_3, $\cdots$, thus forming the trapezoids AP_1, A_1P_2, A_2P_3, $\cdots$. The sum of the areas of these trapezoids will give an approximate value of the area of $APQB$.

$$\text{Area } AP_1 = \tfrac{1}{2}(AP + A_1P_1)\,\Delta x,$$
$$\text{area } A_1P_2 = \tfrac{1}{2}(A_1P_1 + A_2P_2)\,\Delta x,$$
$$\text{area } A_2P_3 = \tfrac{1}{2}(A_2P_2 + A_3P_3)\,\Delta x,$$
$$\text{-------------------------------},$$
$$\text{area } A_{n-1}Q = \tfrac{1}{2}(A_{n-1}P_{n-1} + BQ)\,\Delta x.$$
$$\therefore \text{area of trapezoids} = (\tfrac{1}{2}AP + A_1P_1 + A_2P_2 + \cdots + A_{n-1}P_{n-1} + \tfrac{1}{2}BQ)\,\Delta x.$$

This result may be indicated thus:

$$\textbf{area trapezoids} = \left(\tfrac{1}{2} + 1 + 1 + \cdots + 1 + \tfrac{1}{2}\right)\Delta x,$$

in which the numbers in the brackets are to be taken with the successive ordinates beginning with AP and ending with BQ.

NOTE. It is evident that the greater the number of intervals into which $b - a$ is divided, the more nearly will the total area of the trapezoids come to the actual area between the curve and the x-axis, and, accordingly, the more nearly to the value of the integral. See Exs. 1, 2.

EXAMPLES.

1. Find $\int_1^{12} x^2\,dx$, dividing $12 - 1$ into 11 equal intervals.

Here each interval, Δx, is 1. Hence, approximate value

$= (\tfrac{1}{2}\cdot 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + \tfrac{1}{2}\cdot 12^2) = 577\tfrac{1}{2}$.

The value of $\int_1^{12} x^2\,dx = \left[\frac{x^3}{3} + c\right]_1^{12} = 575\tfrac{2}{3}$. The error in the result obtained by the trapezoidal method is thus, in this instance, less than one-third of one per cent.

2. Show that if 22 equal intervals be taken in the above integral, the approximate value found is 576.125.

3. Show that on using the trapezoidal rule for evaluating $\int_0^{10} x^2\,dx$, if 10 intervals be taken, the result is $1\tfrac{2}{3}$ units more than the true value, and if 20 intervals be taken, the result is $\tfrac{5}{12}$ of a unit more than the true value.

4. Explain why the approximate values found for the integrals in Exs. 1, 2, 3, are *greater* than the true values.

5. Evaluate $\int_{31^\circ}^{32^\circ}\cos x\,dx$ by the trapezoidal rule, taking 10′ intervals.
(*Ans.* .0148. The calculus method gives .0149.)

6. Evaluate $\int_{26^\circ}^{32^\circ}\sin x\,dx$, taking 30′ intervals.
(*Ans.* .0506. Calculus gives .0508.)

7. Evaluate $\int_{25^\circ}^{35^\circ}\cos x\,dx$, taking 1° intervals.
(*Ans.* .1509. Calculus gives .1510.)

125. Parabolic rule* for measuring areas and evaluating definite integrals. Let the area and the integral be as specified in Art. 124. For the application of the parabolic rule, the interval AB is divided into an *even* number of equal intervals each equal to Δx, say. The ordinates are drawn at the points of division. Through each successive set of three points (P, P_1, P_2), (P_2, P_3, P_4), $\cdots$, are drawn arcs of parabolas whose axes are parallel to the ordinates. The area between these parabolic arcs and the x-axis will be approximately equal to the area between the given curve and the x-axis. The area bounded by one of these parabolic arcs and the x-axis, and a pair of ordinates, say the area of the parabolic strip $APP_1P_2A_2$, will now be found.

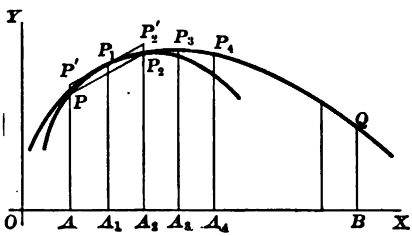

FIG. 64.

$$\text{Parabolic strip } APP_1P_2A_2 = \text{trapezoid } APP_2A_2 + \text{parabolic segment } PP_1P_2. \qquad (1)$$

Now the parabolic segment PP_1P_2

$$= \text{two-thirds of its circumscribing parallelogram } PP'P_2'P_2.\dagger \qquad (2)$$

* This rule, which is much used by engineers for measuring areas, is also known as **Simpson's one-third rule,** from its inventor, Thomas Simpson (1710–1761), Professor of Mathematics at Woolwich.

† See Art. 111, Ex. 19.

Area trapezoid $APP_2A_2 = \frac{1}{2}AA_2(AP + A_2P_2)$;

$$\text{area } PP'P'_2P_2 = \text{area } AP'P'_2A_2 - \text{area } APP_2A_2$$
$$= 2 \cdot \tfrac{1}{2}AA_2 \cdot A_1P_1 - \tfrac{1}{2}AA_2(AP + A_2P_2). \quad (3)$$

Hence, by (1), (2), and (3), area parabolic strip $APP_1P_2A_2$

$$= (AP + 4\,A_1P_1 + A_2P_2)\frac{\Delta x}{3}.$$

Similarly, area of next parabolic strip $A_2P_2P_3P_4A_4$

$$= (A_2P_2 + 4\,A_3P_3 + A_4P_4)\frac{\Delta x}{3};$$

and so on. Addition of the successive areas gives total area of parabolic strip

$$= (AP + 4\,A_1P_1 + 2\,A_2P_2 + 4\,A_3P_3 + 2\,A_4P_4 + \cdots + BQ)\frac{\Delta x}{3}.$$

This result may be indicated thus:

$$\textbf{Total parabolic area} = (1+4+2+4+\cdots+2+4+1)\frac{\Delta x}{3}, \quad (4)$$

in which the numbers in the brackets are understood to be taken with the successive ordinates beginning with AP and ending with BQ.

EXAMPLES.

1. Find $\int_0^{10} x^3\,dx$, taking 10 equal intervals.

Here, each interval = 1. Hence, the result by (4)

$$= (1 \cdot 0^3 + 4 \cdot 1^3 + 2 \cdot 2^3 + 4 \cdot 3^3 + 2 \cdot 4^3 + 4 \cdot 5^3 + 2 \cdot 6^3 + 4 \cdot 7^3 + 2 \cdot 8^3 + 4 \cdot 9^3 + 1 \cdot 10^3) \times \tfrac{1}{3} = 2500.$$

True value $= \left[\frac{x^4}{4} + c\right]_0^{10} = 2500.$

2. Calculate the above integral, using the trapezoidal rule and taking 10 equal intervals.

3. Evaluate $\int_1^{11} x^3\,dx$, both by the trapezoidal and the parabolic rules, taking 10 equal intervals.

4. Evaluate Ex. 1, Art. 124, by the parabolic rule. Why is the result the true value of the integral?

5. Show that there is only an error of $1\frac{1}{3}$ in 20,000 made in evaluating $\int_0^{10} x^4\,dx$ by the parabolic method, when 10 intervals are taken.

6. Find the error in the evaluation of the integral in Ex. 5 by the trapezoidal method, when 10 intervals are taken.

7. Evaluate the integrals in Exs. 6, 7, Art. 124, by the parabolic rule.

NOTE. For a comparison between the trapezoidal and parabolic rules, for a statement of **Durand's rule,** which is an empirical deduction from these two rules, for a statement of **other rules for approximate integration,** and for a note on the outside limits of error in the case of the trapezoidal and parabolic rules, see Murray, *Integral Calculus*, Arts. 86, 87, Appendix, Note E, and foot-note, page 186.

126. Integration in series. The methods described or referred to in Arts. 124 and 125 for evaluating a definite integral $\int_a^b f(x)dx$, give a numerical result only, and do not convey any information as to the anti-differential of $f(x)dx$. Some information, however, about the anti-differential of $f(x)dx$ can be obtained in certain cases by expanding $f(x)$ in a series in ascending or descending powers in x and then integrating $f(x)dx$ term by term. The expanded series can represent $f(x)$ only for the values of x in a certain definite range, namely, the range of values for which the series is convergent. The series obtained by integration is convergent for the same range of values of x, and *for values of x in this range* represents the anti-differential. See **Chapter XIX.** (in particular, Arts. 172, 174), where the question of integration in series is more fully discussed. The following examples and note are given here mainly for the purpose of drawing attention to, and arousing interest in, questions relating to series.

EXAMPLES.

1. Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, show that $\int e^x\,dx = e^x + c$, in which c is a constant.

2. Given that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$, and that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, show that $\int \cos x\,dx = \sin x + c$, and that $\int \sin x\,dx = -\cos x + c$.

3. Do Ex. 2, Art. 174.

4. Find an approximate value of the area of the four-cusped hypocycloid inscribed in a circle of radius 8 inches. (This area can also be found exactly; see Art. 137, Note 5, Ex.)

NOTE. **Expansion of functions in series: (*a*) by differentiation; (*b*) by integration.** For remarks on this topic and for the warrant for the operations in Exs. 5, 6, 7, see Chapter XIX., in particular, Arts. 168 (*e*), 172, 173, 174.

5. Suppose it is known that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$ (1)

and that $$\frac{d}{dx}(\sin x) = \cos x.$$

Differentiation of the members in (1) gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

6. By the binomial theorem,

$$(1 - x^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 - \cdots. \tag{1}$$

Differentiation of each member of (1) and division by x gives

$$(1 - x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \cdots. \tag{2}$$

Result (2) may be verified by expanding $(1 - x^2)^{-\frac{1}{2}}$ by the binomial theorem.

7. Given that $d(\tan^{-1} x) = \frac{1}{1 + x^2}$ (Art. 51) $= 1 - x^2 + x^4 - \cdots$, find $\tan^{-1} x$.

On integration, $\tan^{-1} x + c = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$

From this, on putting $x = 0$, $\tan^{-1} 0 + c = 0$. $\therefore c = -\tan^{-1} 0 = \pm n\pi$, in which n is any integer.

$$\therefore \tan^{-1} x = n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$$

8. Do Exs. 3–5, 8, 9, Art. 174.

127. Mechanical devices for integration. The value of a definite integral may be determined by various instruments. Accordingly, they may be called *mechanical integrators.* Of these there are three classes, viz. *planimeters, integrators,* and *integraphs.* These instruments are a great aid to civil, mechanical, and marine engineers. The area of any plane figure can be easily and accurately calculated by each of these mechanisms. Their right to be termed mechanical *integrators* depends on the facts emphasised in Arts. 96, 98, 123–125; the facts, namely, that a definite integral can be represented by a plane area such that the number of square units in the area is the same as the number of units in the integral, and hence that one way of calculating a definite integral is to make a proper areal representation of the integral and then measure this area.

Planimeters, which are of two kinds, viz. *polar planimeters* and *rolling planimeters,* are designed for finding the area of any plane surface represented by a figure drawn to any scale. The first planimeter was devised in 1814 by J. M. Hermann, a Bavarian engineer. A polar planimeter, which is a development of the planimeter invented by Jacob Amsler at Konigsberg in 1854, is the one most extensively used. By it the area of any figure is obtained by going around the boundary line of the figure with a tracing point and noting the numbers that are indicated on a measuring wheel when the operation of tracing begins and ends.

Integrators and **integraphs** also serve for the measurement of areas; they are adapted, moreover, for making far greater computations and solving more complicated problems, such as the calculation of moments of inertia, centres of gravity, etc. The *integraph* (see Art. 100, Notes 2, 3) is the superior instrument, for it directly and automatically draws the successive integral curves. These give a graphic representation of the integration, and are of great service, especially to naval architects. The measure of an ordinate of the first integral curve, when multiplied by a constant belonging to the instrument, gives a certain area associated with that ordinate (see Art. 100).

NOTE 1. A bicycle with a cyclometer attached may be regarded as a mechanical integrator of a certain kind; for by means of a self-recording apparatus it gives the length of the path passed over by the bicycle.

NOTE 2. Planimeters and integrators are simple, and it is easy to learn to use them.

NOTE 3. A brief account of the *planimeter*, references to the literature on the subject, and a note on the fundamental theory, will be found in Murray, *Integral Calculus*, Art. 88, and Appendix, Note F. Also see Lamb, *Calculus*, Art. 102; Gibson, *Calculus*, § 130. For a fuller account see Henrici, *Report on Planimeters* (Report of Brit. Assoc. for Advancement of Science, 1894, pages 496–523); Hele Shaw, *Mechanical Integrators* (Proc. Institution of Civil Engineers, Vol. 82, 1885, pages 75–143). For references concerning the *integraph* see Art. 100, Note 3.

N.B. Interesting information concerning planimeters, integrators, and the integraph, with good cuts and descriptions, are given in the catalogues of dealers in drawing materials and surveying instruments.

CHAPTER XV.

SUCCESSIVE INTEGRATION. MULTIPLE INTEGRALS. APPLICATIONS.

128. In Chapter VI. (see Arts. 68, 69, 70), successive derivatives and differentials of functions of a single variable were obtained. In Chapter VIII. (see Arts. 79, 80, 82), successive partial derivatives and partial differentials of functions of several variables were discussed. In this chapter processes which are the reverse of the above are performed and are employed in practical applications.

129. Successive integration: One variable. Applications.

Suppose that
$$\int f(x)\,dx = f_1(x), \tag{1}$$

$$\int f_1(x)\,dx = f_2(x), \tag{2}$$

$$\int f_2(x)\,dx = f_3(x). \tag{3}$$

Then, by (3) and (2),
$$f_3(x) = \int\left[\int f_1(x)\,dx\right]dx; \tag{4}$$

By (4) and (1),
$$f_3(x) = \int\left[\int\left(\int f(x)\,dx\right)dx\right]dx. \tag{5}$$

This is written
$$f_3(x) = \int\int\int f(x)(dx)^3,$$

or, more usually,
$$f_3(x) = \int\int\int f(x)\,dx^3. \tag{6}$$

The second member of (6) is called *a triple integral.* Similarly, the second member in (4) is usually written $\int\int f_1(x)\,dx^2$, and is called a double integral.

In general, $\int\int\int\cdots\int f(x)\,dx^n$ denotes the result obtained by

integrating $f(x)\,dx$ n times in succession. This integral is indefinite unless end values of the variable be assigned for each of the successive integrations. This integral and the integrals in (4) and (5) are called *multiple integrals.*

NOTE. It should be observed that here dx^n denotes $dx\,dx\,dx \cdots$ to n factors, *i.e.* $(dx)^n$, and not $d \cdot x^n$ (*i.e.* $nx^{n-1}dx$). [Compare Art. 70.]

EXAMPLES.

1. Find $\int\int\int x^2 dx^3$.

$$\int\int\int x^2 dx^3 = \int\left\{\int\left[\int x^2 dx\right]dx\right\}dx$$
$$= \int\left\{\int\left[\frac{x^3}{3} + c_1\right]dx\right\}dx$$
$$= \int\left\{\frac{x^4}{12} + c_1 x + c_2\right\}dx$$
$$= \frac{x^5}{60} + k_1 x^2 + c_2 x + c_3;$$

for, since c_1 is an arbitrary constant, $\frac{c_1}{2}$ may be denoted by an arbitrary constant k_1.

2. $\int_1^4\int_2^3 x^3 dx^2 = \int_1^4\left[\int_2^3 x^3 dx\right]dx = \int_1^4\left[\frac{x^4}{4} + c\right]_2^3 dx = \frac{65}{4}\int_1^4 dx = \frac{195}{4}.$

3. Determine the curves for every point of which $\frac{d^2y}{dx^2} = 0$. Which of these curves goes through the points (1, 2), (0, 3)? Which of these curves has the slope 2 at the point (3, 5)?

Here $\qquad \frac{d^2y}{dx^2} = 0.$

On integrating, $\qquad \frac{dy}{dx} = c_1.$

On integrating again, $\qquad y = c_1 x + c_2,$
which represents all straight lines.

For the line going through (1, 2) and (0, 3), $2 = c_1 + c_2$ and $3 = 0 + c_2$; whence $c_1 = -1$, $c_2 = 3$. Hence the line is $x + y = 3$.

For the line having the slope 2 at (3, 5), $c_1 = 2$ and $5 = 3c_1 + c_2$, whence $c_2 = -1$. Hence the line is $y = 2x - 1$.

4. Determine the curves for every point of which the second derivative of the ordinate with respect to the abscissa is 6. Which of these curves goes through the points (1, 2), $(-3, 4)$? Which of them has the slope 3 at the point $(-2, 4)$?

N.B. The student is recommended to write sets of data like those in Exs. 3–7, and determine the particular curves that satisfy them. He is also recommended *to draw the curves appearing in these examples.*

5. Determine the curves for every point of which the second derivative of the ordinate with respect to the abscissa is 6 times the number of units in the abscissa. Which of these curves goes through the points (0, 0) (1, 2)? Which of them has the slope 2 at (1, 4)?

6. Determine the curves in which the second derivatives $\frac{d^2y}{dx^2}$ from point to point vary as the abscissas. Find the equation of that one of these curves which passes through (0, 0), (1, 2), (2, 5). Find the equation of that one of these curves which passes through (1, 1), and has the slope 2 at the point (2, 4).

7. Determine the curves in which the second derivative of the abscissa with respect to the ordinate varies as the ordinate. Which of these curves passes through (0, 1), (2, 0), (3, 5)? Which of them has the slope $\frac{1}{2}$ at (1, 2), and passes through (– 1, 3)?

8. A body is projected vertically upward with an initial velocity of 1000 feet per second. Neglecting the resistance of the air and taking the acceleration due to gravitation as 32.2 feet per second, calculate the height to which the body will rise, and the time until it again reaches the ground.

9. Do Ex. 20, Art. 68.

10. When the brakes are put on a train, its velocity suffers a constant retardation. It is found that when a certain train is running 30 miles an hour the brakes will bring it to a dead stop in 2 minutes. If the train is to stop at a station, at what distance from the station should the engineer whistle "down brakes"? (Byerly, *Problems in Differential Calculus.*)

130. Successive integration: several variables. Suppose that

$$\int f(x, y, z)\,dz = f_1(x, y, z), \tag{1}$$

$$\int f_1(x, y, z)\,dy = f_2(x, y, z), \tag{2}$$

$$\int f_2(x, y, z)\,dx = f_3(x, y, z). \tag{3}$$

The integration indicated in (1) is performed as if y and x were constant; the integration in (2) as if x and z were constant; the integration in (3) as if z and y were constant. (Compare Arts. 79, 80.)

From (3) and (2), $f_3(x, y, z) = \int \left\{ \int f_1(x, y, z)\, dy \right\} dx;$ (4)

from (4) and (1), $= \int \left\{ \int \left[\int f(x, y, z)\, dz \right] dy \right\} dx.$ (5)

The second member in (4) is often written

$$\int\int f_1(x, y, z)\, dy\, dx; \tag{6}$$

the second member in (5) is often written

$$\int\int\int f(x, y, z)\, dz\, dy\, dx. \tag{7}$$

The integral in (6) is called a double integral, and the integral in (7) a triple integral.

NOTE 1. It should be observed that according to (2), (3), and (4), integral (6) is obtained by first integrating $f_1(x, y, z)$ with respect to y, and then integrating the result with respect to x; in (7), according to (1), (2), (3), and (5), the first integration is to be made with respect to z, the second with respect to y, and the third with respect to x. That is, the *first integration sign on the right is taken with the first differential on the left, the second integration sign from the right with the second differential from the left*, and so on. When end-values are assigned to the variables, careful attention must be paid to the order in which the successive integrations are performed.

NOTE 2. The notation used above for indicating the order of the variables with respect to which the successive integrations are to be performed, is not universally adopted. Oftentimes, as may be seen by examining various texts on calculus and works which contain applications of the calculus, integrals (6) and (7) are written

$$\int\int f_1(x, y, z)\, dx\, dy, \quad \int\int\int f(x, y, z)\, dx\, dy\, dz \text{ respectively.}$$

In this notation the first integration sign on the right belongs to the first differential on the right, the second integration sign from the right to the second differential from the right, and so on; and the integrations are to be made, first with respect to z, then with respect to y, and then with respect to x. In particular instances, the context will show what notation is employed.

EXAMPLES.

1. $\int\int\int x^2yz^3\, dz\, dy\, dx = \int\int x^2y\left(\frac{z^4}{4} + c_1\right) dy\, dx$

$$= \int x^2\left(\frac{y^2}{2} + c_2\right)\left(\frac{z^4}{4} + c_1\right) dx = \left(\frac{x^3}{3} + c_3\right)\left(\frac{y^2}{2} + c_2\right)\left(\frac{z^4}{4} + c_1\right).$$

2. $\int_2^4\int_1^2\int_2^3 x^2yz^3\,dz\,dy\,dx$ (*i.e.* $\int_{x=2}^{x=4}\int_{y=1}^{y=2}\int_{z=2}^{z=3} x^2yz^3\,dz\,dy\,dx$)

$$=\int_2^4\int_1^2 x^2y\left[\frac{z^4}{4}+c\right]_2^3 dy\,dx=\frac{65}{4}\int_2^4\int_1^2 x^2y\,dy\,dx$$

$$=\frac{65}{4}\int_2^4 x^2\left[\frac{y^2}{2}+c\right]_1^2 dx=\frac{3}{2}\times\frac{65}{4}\int_2^4 x^2dx=455.$$

3. $\int_1^2\int_x^{x^2} x^3y^2\,dy\,dx=\int_{x=1}^{x=2}\left\{\int_{y=x}^{y=x^2} x^3y^2\,dy\right\}dx=\int_1^2 x^3\left[\frac{y^3}{3}+c\right]_x^{x^2}dx$

$$=\tfrac{1}{3}\int_1^2 x^3(x^6-x^3)\,dx=28\tfrac{11}{210}.$$

4. Evaluate the following integrals: (1) $\int_0^2\int_4^3\int_{-2}^1 xy^2z\,dz\,dy\,dx.$

(2) $\int_a^{3a}\int_{\frac{v}{2}}^{\frac{v^2}{a}}(3w-2v)\,dw\,dv.$ (3) $\int_0^a\int_t^{10t}\sqrt{st-t^2}\,ds\,dt.$

(4) $\int_0^\pi\int_0^{a(1-\cos\theta)} r^2\cos\theta\,dr\,d\theta.$ (5) $\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx.$

(6) $\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\phi\,d\theta.$ (7) $\int_0^{2a}\int_0^{\cos^{-1}\left(\frac{r}{2a}\right)} r\,d\theta\,dr.$

(8) $\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\,dr\,d\theta.$ (9) $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\sqrt{a^2-r^2}\cdot r\,dr\,d\theta.$

131. Application of successive integration to finding areas: rectangular coördinates.

EXAMPLES.

1. Find the area between the curve $y^2=8x$, the x-axis, and the ordinate for which $x=3$.

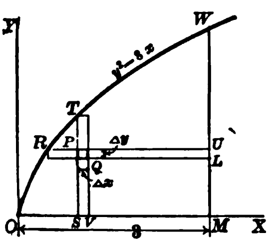

FIG. 65.

At P, any point within the figure OWM whose area is required, suppose that a rectangle PQ having infinitesimal sides dx and dy parallel to the axis is constructed. The area OWM is the limit of the sum of all rectangles such as PQ which can be constructed side by side in OWM. Let one of the vertical sides of the rectangle be produced both ways until it meets the curve and the x-axis in T and S; complete the rectangle TV as in the figure.

First, find the area of the rectangular strip TV by finding the limit of the sum of the rectangles PQ inscribed in it from S to T; then find the limit of the sum of the strips like TV which can be inserted between OY and MW.

$$\text{Area } TV = \lim \sum_{y \text{ at } S}^{y \text{ at } T} (\text{rectangles } PQ) = \int_{y=0}^{y=\sqrt{8x}} dy\, dx = \sqrt{8\,x}\, dx. \tag{1}$$

$$\text{Area } OMW = \lim \sum_{x \text{ at } O}^{x \text{ at } M} (\text{strips } TV) = \int_{x=0}^{x=3} \left[\int_{y=0}^{y=\sqrt{8x}} dy \right] dx \tag{2}$$

$$= 2\sqrt{2} \int_0^3 x^{\frac{1}{2}}\, dx = 4\sqrt{6} \text{ square units.}$$

The last expression in (2) is usually written $\int_0^3 \int_0^{\sqrt{8x}} dy\, dx.$

The area of OWM may also be found by finding the limit of the sum of the rectangles PQ which may be inserted between R and U, and then finding the limit of the sum of the strips like RL which may be inserted between OM and W. Thus,

$$\text{area } RL = \int_{x \text{ at } R}^{x \text{ at } L} dx\, dy = \int_{\frac{y^2}{8}}^{3} dx\, dy = \left(3 - \frac{y^2}{8}\right) dy\,; \tag{3}$$

$$\text{area } OMP = \int_{y \text{ at } M}^{y \text{ at } W} \left(3 - \frac{y^2}{8}\right) dy = \int_0^{\sqrt{24}} \left(3 - \frac{y^2}{8}\right) dy = 4\sqrt{6}. \tag{4}$$

From (3) and (4), $\quad \text{area } OMP = \int_0^{\sqrt{24}} \int_{\frac{y^2}{8}}^{3} dx\, dy.$

NOTE 1. The last expression in (1) is $y\, dx$, the element of area employed in Art. 96.

NOTE 2. Ex. 1 has been solved as above merely in order to give a practical application of double integration.

NOTE 3. For finding areas by double integration in the case of polar coördinates, see Art. 136, Note 3.

2. Express some of the areas in Art. 111 by double integrals, and perform the integrations.

3. Find by double integration the area included between the parabolas $3\,y^2 = 25\,x$ and $5\,x^2 = 9\,y$. [See Murray, *Integral Calculus*, Art. 61, Ex. 1.]

132. Application of successive integration to finding volumes: rectangular coördinates.

EXAMPLES.

1. Find the volume bounded by the surface whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Fig. O-ABC represents one-eighth of the volume required. Suppose that an infinitesimal parallelopiped P_1Q_3 is taken at $P_1(x,\, y,\, o)$, having infinitesi-

mal sides dx, dy, dz, parallel to the x-, y-, and z-axes, respectively. The volume of O–ABC is the limit of the sum of all infinitesimal parallelopipeds such as P_1Q_3 which can be enclosed by OBA, OAC, OCB, and the curvi-

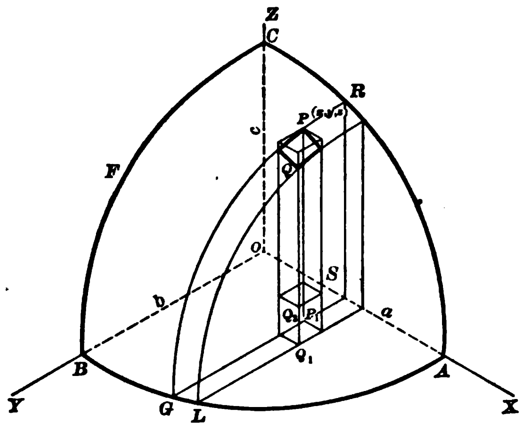

FIG. 66.

linear surface ABC. Construct a parallelopiped PQ_1 by producing the vertical faces of P_1Q_3 to the height P_1P. (The point $P(x, y, z)$ is taken on the surface ABC.)

$$\text{Vol. } PQ_1 = \int_{z \text{ at } P_1}^{z \text{ at } P} dz\,dy\,dx = \left[\int_{z=0}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\right] dy\,dx. \qquad (1)$$

NOTE 1. The numbers x and y are constant along P_1P, and, accordingly, in the integration of (1) x and y are treated as constants.

Now take a slice RGL the planes of whose faces coincide with two faces of PQ_1, as shown in the figure.

Vol. slice $RPGLS$ = limit of sum of parallelopipeds PQ_1 from S to G.

That is, $$\text{vol. slice } RG = \int_{y \text{ at } S}^{y \text{ at } G}\left[\int_{z=0}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\right] dy \cdot dx$$

$$= \left\{\int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}}\left[\int_{z=0}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\right] dy\right\} dx. \qquad (2)$$

NOTE 2. The number x is constant along SG, and, accordingly, in the integration of (2) x is treated as a constant.

Now find the limit of the sum of all infinitesimal slices like RGL from OCB to A; *i.e.* from $x = 0$ to $x = a$. This limit is the volume of O–ABC.

$$\therefore \text{ vol. } O\text{–}ABC = \int_{x \text{ at } O}^{x \text{ at } A} \left\{ \int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} \left[\int_{z=0}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz \right] dy \right\} dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx. \qquad (3)$$

On performing the integrations indicated in (3) (see Ex. 4 (5), Art. 130), it will be found that

$$\text{vol. } O\text{–}ABC = \tfrac{1}{6}\pi abc. \quad \text{Hence vol. ellipsoid} = \tfrac{4}{3}\pi abc.$$

NOTE 3. Result (3) may be written $\int_{x \text{ at } O}^{x \text{ at } A} \int_{y \text{ at } S}^{y \text{ at } G} \int_{z \text{ at } P_1}^{z \text{ at } P} dx\, dy\, dz.$

NOTE 4. The initial element of volume P_1Q_3, *i.e.* $dx\, dy\, dz$, is an infinitesimal of the third order; the parallelopiped PQ_1 is an infinitesimal of the second order; the slice RGL is an infinitesimal of the first order.

NOTE 5. Equally well, slices may be taken which are parallel to the xz-plane or to the yz-plane.

NOTE 6. Instead of the parallelopiped PQ_1, equally well, a similar parallelopiped can be taken whose finite edges are parallel to the y-axis, or to the x-axis.

2. Perform the integrations indicated in Ex. 1.

3. Do Ex. 1 by taking the elements in the ways indicated in Notes 5 and 6.

4. From the result in Ex. 1 deduce the volume of a sphere of radius a. Also deduce the volume of this sphere by the method used in Ex. 1. (Compare with the methods used in Art. 112, Ex. 19 and Note 3.)

5. Two cuts are made across a circular cylindrical log which is 20 inches in diameter; one cut is at right angles to the axis of the cylinder, the other cut makes an angle of 60° with the first cut, and both cuts intersect the axis of the cylinder at the same point. Find the volume of each of the wedges thus obtained.

6. As in Ex. 5, for the general case in which the radius of the log is a and the angle between the cuts is α. Thence deduce the result in Ex. 5.

7. The centre of a sphere of radius a is on the surface of a right cylinder the radius of whose base is $\frac{a}{2}$. Find the volume of the part of the cylinder intercepted by the sphere.

8. Taking the same conditions as in Exs. 5, 6, excepting that the cuts intersect on the surface of the log, find the volume intercepted between the cuts.

133. Application of successive integration to finding volumes; polar coördinates.

A. The use of polar coördinates in finding volumes sometimes leads to easier integrations than does the use of rectangular coördinates.

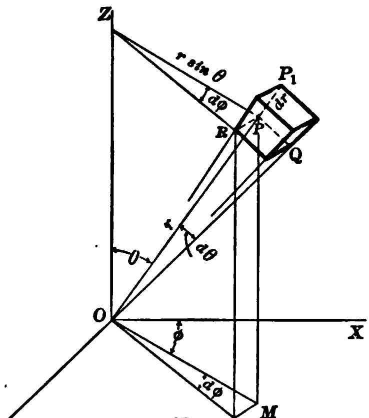

FIG. 67.

Let O, the origin, be taken as pole. The infinitesimal element of volume is formed as follows: Take any point $P(r, \theta, \phi)$. [Here $r = OP$, $\theta =$ angle POZ, $\phi =$ angle XOM, OM being the projection of OP on XOY. In other words, $\phi =$ the angle between the plane XOZ and the vertical plane in which OP lies.] Produce OP an infinitesimal distance dr to P_1, and revolve OP_1 through an infinitesimal angle $d\theta$ in the plane ZOP to the position OQ. Now revolve OPP_1Q about OZ through an infinitesimal angle $d\phi$, keeping θ constant. The solid PP_1QR is thus generated. Its edges PP_1, PQ, PR are respectively dr, $r\,d\theta$, $r \sin\theta\, d\phi$; its volume (to within an infinitesimal of an order lower than the third) is $r^2 \sin\theta\, dr\, d\phi\, d\theta$. On determining the proper limits for r, ϕ, θ, and integrating, the volume required is obtained.

Ex. 1. Find the volume of a sphere of radius a, using polar coördinates and taking O on the surface of the sphere and OZ on the diameter through O. (It will be found that the volume is given by the integral in Art. 130, Ex. 4, (6). See Murray, *Integral Calculus*, Art. 63, Ex. 1.)

B. The element of volume can be chosen in another way, which sometimes leads to simpler integrations than are otherwise obtainable. An instance is given in Ex. 2 below.

EXAMPLES.

2. Another way of doing Ex. 7, Art. 132.

In the figure, O–ABC is one-eighth the sphere, and the solid bounded by the plane faces $ALBO$, AKO, the spherical face $ALBVA$, and the cylindrical

face $AVBOKA$ is one-fourth of the part of the cylinder intercepted by the sphere.

In AOK take any point P. Let $OP = r$, and angle $AOP = \theta$. Produce OP an infinitesimal distance dr to P_1, and revolve OPP_1 through an infinitesimal angle $d\theta$. Then PP_1 generates a figure, two of whose sides are dr and $r\,d\theta$. Its area (to within an infinitesimal of an order lower than the second) is $r\,dr\,d\theta$. (See Art. 136, Note 3, Ex. 8.)

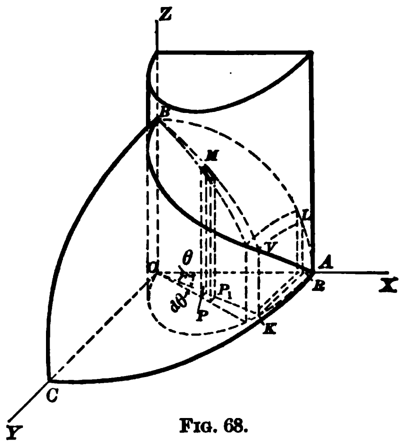

FIG. 68.

On this infinitesimal area as a base, erect a vertical column to meet the sphere in M. Then $PM = \sqrt{a^2 - r^2}$, and the volume of the column is $\sqrt{a^2 - r^2} \cdot r\,dr\,d\theta$. This is taken as the element of volume; the limit of the sum of these columns standing on AOK is the volume required. Keeping θ constant, first find the limit of the sum of the columns standing on the sector extending from O to K whose angle is $d\theta$. Since $OK = a\cos\theta$, this limit is $\int_{r=0}^{r=a\cos\theta} \sqrt{a^2 - r^2} \cdot r\,dr\,d\theta$. This gives the volume of a wedge-shaped slice whose thin edge is OB. One-fourth of the volume required is the limit of the sum of all the wedge-shaped slices of this kind that can be inserted between AOB and COB; that is, from $\theta = 0$ to $\theta = \frac{\pi}{2}$; *i.e.*

$$\text{vol. required} = 4\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=a\cos\theta} \sqrt{a^2 - r^2} \cdot r\,dr\,d\theta = \tfrac{2}{3}\pi a^3 - \tfrac{8}{9}a^3.$$

[See Art. 130, Ex. 4 (9).]

In this instance this is a very much shorter way of deriving the volume than by starting with the element $dx\,dy\,dz$, as in Art. 132.

3. Find the volume of a sphere of radius a, taking O at the centre: (1) choosing the element of volume as in A; (2) choosing it as in B.

4. The axis of a right circular cylinder of radius b passes through the centre of a sphere of radius a $(a > b)$. Find the volume of that portion of the sphere which is external to the cylinder.

CHAPTER XVI.

FURTHER GEOMETRICAL APPLICATIONS OF INTEGRATION.

134. In this chapter the calculus is used for finding volumes in a particular case, for finding areas of curves whose equations are given in polar coördinates, for finding the lengths of curves whose equations are given either in rectangular or in polar coördinates, for finding the areas of surfaces in two special cases, and for finding mean values of variable quantities.

N.B. Many of the problems in this chapter are presented in a general form. In such cases the student is recommended, when he obtains the general result, to make immediate application of it to particular concrete cases.

135. Volumes of solids the areas of whose cross-sections can be expressed in terms of one variable. In Art. 112 the volumes of solids of revolution were found by making cross-sections of the solid at right angles to the axis of revolution, taking these cross-sections an infinitesimal distance apart, and finding the limit of the sum of the infinitesimal slices into which the solid is thus divided. This method of finding the volume of a solid can sometimes be easily applied in the case of solids which are not solids of revolution. The general method is: (a) to take a cross-section in some convenient way; (b) to express the area of this cross-section in terms of some variable; (c) to take a parallel cross-section at an infinitesimal distance from the first cross-section; (d) to express the volume of the infinitesimal slice thus formed, in terms of the variable used in (b); (e) to find the limit of the sum of the infinite number of like parallel slices into which the solid can thus be divided. There is often occasion for the exercise of judgment in taking the cross-sections conveniently.

EXAMPLES.

1. Find the volume of a right conoid with a circular base of radius a and an altitude h.

NOTE 1. A conoid is a surface which may be generated by a straight line which moves in such a manner as to intersect a given straight line and a given curve and always be parallel to a given plane. In the conoid in this example the given plane is at right angles to the given straight line, and the perpendicular erected at the centre of the circle to the plane of the base intersects the given straight line.

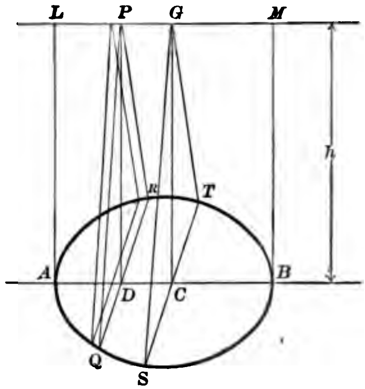

FIG. 69.

Let LM be the fixed line and ARB the fixed circle having its centre at C. Take a cross-section PQR at right angles to LM, and, accordingly, at right angles to a diameter AB. Let it intersect AB in D, and denote CD by x.

Area

$$PQR = \tfrac{1}{2}PD \cdot QR = PD \cdot QD.$$

Now $PD = h$, and, by elementary geometry,

$$QD = \sqrt{AD \cdot DB} = \sqrt{(a-x)(a+x)} = \sqrt{a^2 - x^2}.$$

$$\therefore \text{ area } PQR = h\sqrt{a^2 - x^2}.$$

Now take a cross-section parallel to PQR at an infinitesimal distance from it. Since CD has been denoted by x, this infinitesimal distance may be denoted by dx.

$$\begin{aligned}\text{Vol. } LM\text{–}BQARB &= 2 \text{ vol. } LG\text{–}TSAT \\ &= 2 \lim_{x \text{ at } C}^{x \text{ at } A} (\text{sum of slices } PQR) \\ &= 2h\int^a \sqrt{a^2 - x^2}\,dx = \tfrac{1}{2}\pi a^2 h.\end{aligned}$$

That is, the volume of the conoid is one-half the volume of a cylinder of radius a and height h. (See Echols, *Calculus*, Ex. 3, p. 266.)

NOTE 2. As already observed, finding the volumes of solids of revolution is a special case under this article.

NOTE 3. *Two general methods of finding volumes* have now been shown, namely, the method shown in Arts. 132, 133, and the method shown in this article.

2. Do Ex. 1, denoting AD by x.

3. Do Ex. 8, Art. 112 and Ex. 1, Art. 132 by method of this article.

4. Find the volume of a right conoid of height 8 which has an elliptic base having semi-axes 6 and 4, and in which the fixed line is parallel to the major axis. Find the volume in the general case in which the height is h, the semi-major axis a, and the semi-minor axis b.

5. A rectangle moves from a fixed point, one side varying as the distance from the point, and the other side as the square of this distance. At the distance of 3 feet the rectangle is a square whose side is 5 feet. What is the volume generated when the rectangle moves from the distance 2 feet to the distance 4 feet?

6. On the double ordinates of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, and in planes perpendicular to that of the ellipse, isosceles triangles having vertical angles 2α are erected. Find the volume of the surface thus generated.

7. A circle of radius a moves with its centre on the circumference of an equal circle, and keeps parallel to a given plane which is perpendicular to the plane of the given circle: find the volume of the solid thus generated.

8. Two cylinders of equal altitude h have a circle of radius a for their common upper base. Their lower bases are tangent to each other. Find the volume common to the two cylinders.

136. Areas: polar coördinates. Suppose there is required the area of the figure bounded by the curve whose equation is $f(r, \theta) = 0$, and the radii vectores drawn to two assigned points on this curve.

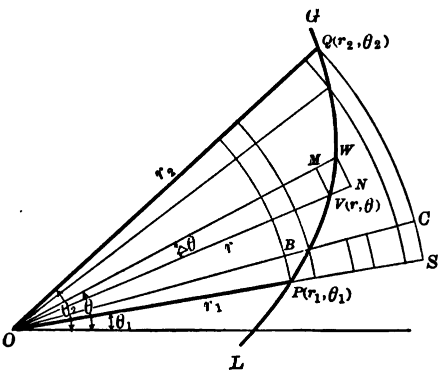

FIG. 70.

Let LG be the curve $f(r, \theta) = 0$, and P and Q the points (r_1, θ_1) and (r_2, θ_2) respectively; it is required to find the area POQ. Suppose that the angle POQ is divided into n equal angles each equal to $\Delta\theta$, and let VOW be one of these angles. Denote V as the point (r, θ). Through V, about O as a centre, draw a circular arc intersecting OW in M.

Through W, about O as a centre, draw a circular arc intersecting OV in N. Denote MW by Δr.

Then, area $OVM = \frac{1}{2} r^2 \Delta\theta$ (*Pl. Trig.*, p. 175), and area $ONW = \frac{1}{2}(r + \Delta r)^2 \Delta\theta$.

Let "inner" and "outer" circular sectors, like VOM and NOW in the case of VW, be formed for each of the arcs like VW which are subtended by angles equal to $\Delta\theta$ and lie between P and Q. It is evident that

$$\text{total area of inner sectors} < \text{area } POQ < \text{total area of outer sectors}. \quad (1)$$

In the case of the arc VW the difference between the inner and outer sectors is $VMWN$. On noting this difference for each arc and transferring it to the radius vector OPS, as indicated in the figure, it is apparent that the total difference between the areas of the inner and outer sectors is $PBCS$. Now

$$\text{area } PBCS = \text{area } OSC - \text{area } OPB = \tfrac{1}{2}(\overline{OS}^2 - \overline{OP}^2)\Delta\theta;$$

and this approaches zero when $\Delta\theta$ approaches zero.

From these facts and relation (1) it follows that

Area POQ = limit of area of inner sectors (or outer sectors) when $\Delta\theta$ approaches zero, that is, when the number of these sectors becomes infinitely great. That is,

Area POQ = limit of sum of areas of sectors VOM from OP to OQ when $\Delta\theta$ approaches zero

$$= \lim_{\Delta\theta \doteq 0} \sum_{\theta=\theta_1}^{\theta=\theta_2} \tfrac{1}{2} r^2 \Delta\theta = \tfrac{1}{2} \int_{\theta_1}^{\theta_2} r^2\, d\theta. \qquad \text{(See Art. 96.)}$$

NOTE 1. The element of area in polar coördinates is thus $\frac{1}{2} r^2 d\theta$; this is the area of an infinitesimal circular sector, of which the radius is r and the angle is an infinitesimal, $d\theta$. The differential of the area also has the same form $\frac{1}{2} r^2 d\theta$. In the element of area $d\theta$ must be infinitesimal, in the differential $d\theta$ need not be infinitesimal. (See Art. 67 *b*.)

NOTE 2. It is not necessary that the angles $\Delta\theta$ be all equal. (See Art. 96, Note 3.)

EXAMPLES.

1. Find the area of a loop of the curve $r = a \sin 2\theta$.

It is first necessary to find the values of θ at the beginning and at the end of a loop. At O (see Fig., page 414) $r = 0$; hence, $\sin 2\theta = 0$ at O. If $\sin 2\theta = 0$, then $2\theta = 0,\ \pi,\ 2\pi,\ \cdots$, and, accordingly, $\theta = 0,\ \frac{\pi}{2},\ \pi,\ \cdots$. Any pair of consecutive values, say 0 and $\frac{\pi}{2}$, are values of θ at O at the beginning and end of a loop.

$$\therefore \text{area of a loop} = \tfrac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \sin^2 2\theta = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \tfrac{1}{8}\pi a^2.$$

2. Find the area of one of the loops of the curve $r = a \sin 3\theta$.

3. Find (1) the area of a loop of the lemniscate $r^2 = a^2 \cos 2\theta$; (2) the area of a loop of the curve $r^2 = a^2 \cos n\theta$.

4. Show that (1) the area included between the hyperbolic spiral $r\theta = a$ and any two radii vectores is proportional to the difference between the lengths of these radii vectores; (2) the area included between the logarithmic spiral $r = e^{a\theta}$ and any two radii vectores is proportional to the difference between the squares on these radii vectores.

5. Find the area enclosed by the cardioid $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{\theta}{2}$.

6. Find the area of the oval $r = 3 + 2\cos\theta$.

7. Compute the area of the loop of the folium of Descartes $x^3 + y^3 = 3axy$.

SUGGESTION for Ex. 7: Change to polar coördinates, and then use the substitution $z = \tan\theta$.

Note 8. *On finding areas of curves by double integration.* For the sake of illustration an example will be shown in which areas, in polar coördinates, are found by double integration.

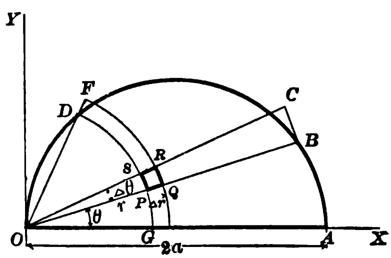

FIG. 71.

8. Find the area of the circle $r = 2a\cos\theta$.

Take any point P in ODA. Let $OP = r$, angle $AOP = \theta$. Produce OP a distance Δr to Q; revolve OPQ through an angle $\Delta\theta$. Then PQ sweeps over the area $PQRS$.

Area $PQRS$

$$= \tfrac{1}{2}\,\overline{OQ}^2 \cdot \Delta\theta - \tfrac{1}{2}\,\overline{OP}^2 \cdot \Delta\theta$$

$$= r \cdot \Delta r \cdot \Delta\theta + \tfrac{1}{2}(\Delta r)^2 \cdot \Delta\theta.$$

One can proceed to find the limit of the sum of the areas like $PQRS$ in ODA, in either of the two following ways (*a*) and (*b*).

(*a*) Starting with $PQRS$ as an element of area, find the area of the sector BOC; then, using BOC as an element of area, derive therefrom the area of ODA. Thus,

$$\text{area } BOC = \lim_{\Delta r \doteq 0} \sum_{r=0}^{r=OB} PQRS = \int_{r=0}^{r=2a\cos\theta} r\,dr \cdot \Delta\theta;$$

$$\text{area } ODA = \lim_{\Delta\theta \doteq 0} \sum_{\theta=0}^{\theta=AOY} BOC \quad = \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r\,dr\,d\theta = \frac{\pi a^2}{2}.$$

(*b*) Starting with $PQRS$ as an element of area, find the area of the circular strip GDF; then using GDF as an element of area, derive therefrom the area of ODA. Thus,

$$\text{area } GDF = \lim_{\Delta\theta \doteq 0} \sum_{\theta=0}^{\theta=\cos^{-1}\left(\frac{OD}{OA}\right)} PQRS = \int_{\theta=0}^{\theta=\cos^{-1}\left(\frac{r}{2a}\right)} r\,d\theta \cdot \Delta r;$$

$$\text{area } ODA = \lim_{\Delta r \doteq 0} \sum_{r=0}^{r=OA} GDF \qquad = \int_0^{2a} \int_0^{\cos^{-1}\left(\frac{r}{2a}\right)} r\,d\theta\,dr = \frac{\pi a^2}{2}.$$

$\therefore$ area of circle $= 2$ area $ODA = \pi a^2$. [Ex. 4 (7), Art. 130.]

In this method of computing areas the infinitesimal element of area is thus $r\,dr\,d\theta$.

Note 4. For discussions on the sign to be given to an area, on the areas of closed curves, and on the area swept over by a moving line, see Lamb, *Calculus*, Arts. 99, 101; Gibson, *Calculus*, §§ 128, 129; Echols, *Calculus*, Arts. 163, 164.

137. Lengths of curves: rectangular coördinates. Let it be required to find the length of an arc PQ of the curve whose equation is $y = f(x)$, or $F(x, y) = 0$. Let P, Q be the points (x_1, y_1), (x_2, y_2) respectively, and denote the length of PQ by s.

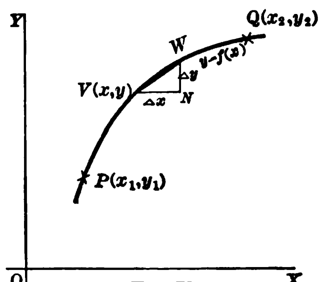

FIG. 72.

Suppose that chords like VW are inscribed in the arc from P to Q. Through V draw VN parallel to the x-axis, and through W draw

WN parallel to the y-axis. Let V be (x, y) and W be $(x + \Delta x, y + \Delta y)$. Then $VN = \Delta x$, $WN = \Delta y$, and

$$\text{chord } VW = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{1}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x \quad (2) \quad = \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2} \cdot \Delta y. \quad (3)$$

Now suppose that Δx, and consequently Δy, approach zero; then the arc VW and the chord VW both become infinitesimal. The smaller the chords VW from P to Q are taken, the more nearly will their sum approach to the length of the arc PQ. The difference between their sum and the length of PQ can be made as small as one pleases, simply by decreasing the arcs. Thus:

s = limit of sum of chords VW when these chords become infinitesimal *

$$= \lim_{\Delta x \doteq 0} \sum_{x=x_1}^{x=x_2} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx. \quad \text{(Definitions, Arts. 22, 23, 96.)} \quad (4)$$

Similarly, from form (3),

$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy. \tag{5}$$

NOTE 1. The quantities under the integration sign in (4) and (5) are the infinitesimal elements of length in rectangular coördinates. The differential of the arc also has the same forms (Art. 67 c); see Note 1, Art. 136.

NOTE 2. In (4) the integrand must be expressed in terms of x; in (5) in terms of y.

NOTE 3. The process of finding the length of a curve is often called *the rectification of the curve;* for it is equivalent to getting a straight line of the same length as the curve. †

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* For rigorous proof of this, depending on elementary algebra and geometry, see Rouché et Comberousse, *Traité de Géométrie* (1891), Part I., § 291. For a proof of the same principle and for interesting remarks on the length and rectification of a curve, see Echols, *Calculus*, Arts. 165, 172.

† The semi-cubical parabola was the first curve that was ever rectified absolutely. William Neil (1637–1670), a pupil of Wallis at Oxford, found the length of any arc of this curve in 1657. This was also accomplished

NOTE 4. It has been pointed out in Art. 19, Ex. 6, Note, that the difference between an infinitesimal arc and its chord is an infinitesimal of an order at least three lower. From this and Theorems A and B, Art. 21, it follows that the limit of the sum of an infinite number of infinitesimal arcs is the same as the limit of the sum of the chords of these arcs.

EXAMPLES.

1. Find the length of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$\text{Length of a quadrant} = \int_{x=0}^{x=a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{1}$$

On differentiation, $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$; whence $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$.

$$\therefore \text{ a quadrant} = \int_0^a \sqrt{1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx = \int_0^a \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx = \int_0^a \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}\, dx = \frac{3}{2}a.$$

$\therefore$ length of hypocycloid $= 4 \times \frac{3}{2}a = 6a$.

NOTE 5. The hypocycloid, sometimes called the astroid, may also be represented by the equations $x = a\cos^3\theta$, $y = a\sin^3\theta$. (This may be verified by substitution.) On using these equations it follows that

$$dx = -3a\cos^2\theta\sin\theta\, d\theta,\ dy = 3a\sin^2\theta\cos\theta\, d\theta,$$

whence
$$\frac{dy}{dx} = -\tan\theta.$$

Thence (1) becomes:

$$\text{length of quadrant} = -\int_{\theta=\frac{\pi}{2}}^{\theta=0} \sqrt{1 + \tan^2\theta} \cdot 3a\cos^2\theta\sin\theta\, d\theta$$

$$= 3a\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta = \frac{3a}{2}, \text{ as before.}$$

(Ex. Show that the area of the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$ is $\frac{3}{8}\pi a^2$; and that the volume generated by its revolution about the x-axis is $\frac{32}{105}\pi a^3$, as obtained otherwise in Art. 112, Ex. 20.)

2. Find the lengths of the following:

(1) The circle $x^2 + y^2 = a^2$. (2) The arc of the parabola $y^2 = 4ax$, (a) from the vertex to the point (x_1, y_1); (b) from the vertex to the end of the latus

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independently by Heinrich van Heuraet in Holland. The second curve to be rectified was the cycloid. This was effected by the famous architect, Sir Christopher Wren (1632–1723), in 1673, and also by the French mathematician, Pierre de Fermat (1601–1665).

rectum. (3) (*a*) The arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $\theta = \theta_0$ to $\theta = \theta_1$; (*b*) a complete arch of this cycloid. (4) The arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, (*a*) from the vertex to (x_1, y_1); (*b*) from the vertex to the point for which $x = a$.

3. Find the whole length of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$. Thence deduce the length of the hypocycloid.

4. Show that in the ellipse $x = a\sin\phi$, $y = b\cos\phi$, ϕ being the complement of the eccentric angle, the arc s measured from the extremity of the minor axis is $a\int_0^{\phi}\sqrt{1-e^2\sin^2\phi}\,d\phi$, e being the eccentricity. (This integral is called "the elliptic integral of the second kind.") Then show that the perimeter of an ellipse of small eccentricity e is approximately $2\pi a\left(1 - \frac{e^2}{4}\right)$.

138. Lengths of curves: polar coördinates. Let it be required to find the length of an arc PQ of the curve $f(r, \theta) = 0$. Let P and Q be the points (r_1, θ_1), (r_2, θ_2), respectively, and denote the length of arc PQ by s. Suppose that chords like VW are inscribed in the arc from P to Q. Let V and W be denoted as the points (r, θ), $(r + \Delta r, \theta + \Delta\theta)$, respectively. Then, from Eq. (2) Art. 67 *d*,

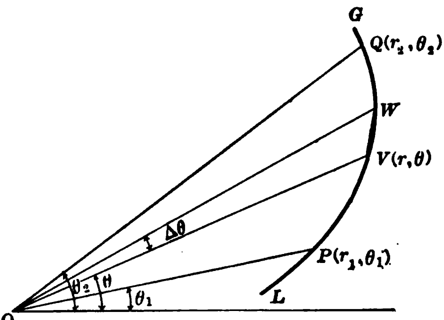

FIG. 73.

$$\text{chord } VW = \sqrt{\left(r\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(r\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\cdot\sin\tfrac{1}{2}\Delta\theta + \frac{\Delta r}{\Delta\theta}\right)^2}\cdot\Delta\theta. \quad (1)$$

The length of the arc PQ (see Art. 137) is the limit of the sum of the lengths of the chords VW from P to Q, when these chords become infinitesimal, that is when $\Delta\theta$ approaches zero. Hence, from (1) and the definitions of a derivative and an integral,

$$s = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\cdot d\theta. \qquad (2)$$

It can also be shown [see the derivation of result (6), Art. 67 *d*],

that $$s=\int_{r_1}^{r_2}\sqrt{r^2\left(\frac{d\theta}{dr}\right)^2+1}\cdot dr. \tag{3}$$

NOTE 1. The quantities under the integration sign in (2) and (3) are the infinitesimal elements of length in polar coördinates. The differential of the arc also has the same forms, Art. 67 *d*; see Note 1, Art. 137.

NOTE 2. In (2) the integrand must be expressed in terms of θ; in (3), in terms of r.

NOTE 3. **The intrinsic equation of a curve.** See Appendix, Note B.

EXAMPLES.

1. Find the length of the cardioid $r=a(1-\cos\theta)$.

Here $$s=2\int_{\theta=0}^{\theta=\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

The substitution of the value of r and $\frac{dr}{d\theta}$ in the integrand and simplification, give

$$s=2\,a\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\,d\theta=4\,a\int_0^{\pi}\sin\frac{\theta}{2}\,d\theta=8\,a.$$

2. Find the lengths of the following:

(1) The circle $r=a$. (2) The circle $r=2\,a\sin\theta$. (3) The curve $r=a\sin^3\frac{\theta}{3}$. (4) The arc of the equiangular spiral $r=ae^{\theta\cot\alpha}$, (*a*) from $\theta=0$ to $\theta=2\pi$; (*b*) from $\theta=2\pi$ to $\theta=4\pi$. (5) The arc of the spiral of Archimedes $r=a\theta$ from $(r_1,\ \theta_1)$ to $(r_2,\ \theta_2)$. (6) The arc of the parabola $r=a\sec^2\frac{\theta}{2}$, (*a*) from $\theta=0$ to $\theta=\theta_1$; (*b*) from $\theta=-\frac{\pi}{2}$ to $\theta=+\frac{\pi}{2}$.

139. Areas of surfaces of revolution.

NOTE 1. GEOMETRICAL THEOREM. Let KL and RS (Fig. 74 *a*) be in the same plane. In elementary solid geometry it is shown that if a finite straight line KL makes a complete revolution about RS, the surface thus generated by KL is equal to $2\pi TM\cdot KL$, in which TM is the length of the perpendicular let fall on RS from T, the middle point of KL.

Suppose that an arc PQ of a curve $y=f(x)$ revolves about the x-axis, and that the area of the surface thus generated is required.

Let P and Q be the points (x_1, y_1) and (x_2, y_2) respectively. Suppose that PQ is divided into small arcs such as KL, and denote K and L as the points (x, y) and $(x + \Delta x, y + \Delta y)$ respectively.

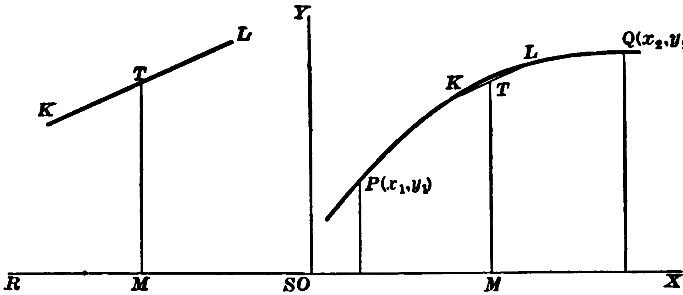

FIG. 74 *a*. FIG. 74 *b*.

Draw the chord KL, and from T, the middle point of this chord, draw TM at right angles to the x-axis. Then *the area generated by the chord KL when the arc PQ revolves about the x-axis*

$$= 2\pi TM \cdot KL$$

$$= 2\pi(y + \tfrac{1}{2}\Delta y)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x. \qquad \text{(Note 1.)}$$

The smaller the chords KL are taken, the more nearly will the surfaces generated by them approach coincidence with the surface generated by the arc PQ, and the difference between area of the latter surface and the sum of the areas of the former surfaces can be made as small as one pleases by decreasing Δx. Accordingly, the area of the surface generated by the arc PQ is the limiting value of the sum of the areas of the surfaces generated by the chords KL (from P to Q) when these chords become infinitesimal. That is, **area of surface generated by PQ**

$$= \lim_{\Delta x \doteq 0} \sum_{x=x_1}^{x=x_2} 2\pi(y + \tfrac{1}{2}\Delta y)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x \qquad (1)$$

$$= 2\pi \int_{x_1}^{x_2} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad \text{(Definitions of derivative and integral.)} \qquad (2)$$

If the length of the chord KL be denoted by $\sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}\Delta y$, this integral takes the form

$$\text{surface} = 2\pi\int_{y_1}^{y_2} y\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy. \qquad (3)$$

NOTE 2. Each of the expressions to be integrated in (2) and (3) may be denoted by $2\pi y\,ds$ [Art. 67 f(9)], and is called an element of the surface of revolution.

If PQ is revolved about the y-axis, the element of surface is $2\pi x\,ds$; and the surface

$$= 2\pi\int_{x=x_1,\,y=y_1}^{x=x_2,\,y=y_2} x\,ds = 2\pi\int_{y=y_1}^{y=y_2} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$$

$$= 2\pi\int_{x=x_1}^{x=x_2} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx. \qquad (4)$$

The questions, whether to use form (2) or (3), and which of (4) to employ, are decided by convenience and ease of working. (See Art. 136, Note 1, and Art. 67 f.)

NOTE 3. In a similar manner it can be shown that the area of the surface generated by the revolution of an arc of a curve about any straight line in the plane of the arc, is

$$2\pi\int_{e_1}^{e_2} l\,ds, \qquad (5)$$

in which ds denotes an infinitesimal arc of the curve, l the distance of this infinitesimal arc from the straight line, and e_1 and e_2 are coördinates of some kind that denote the ends of the revolving arc. An illustration is given in Ex. 4.

EXAMPLES.

1. Find the surface generated by the revolution of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

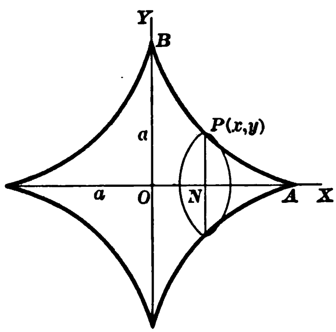

FIG. 75.

Surface

$$= 2\int_{x=0}^{x=a} 2\pi\cdot PN\cdot ds$$

$$= 4\pi\int_{x=0}^{x=a} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$= 4\pi\int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}\cdot\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}dx$$

(See Art. 137, Ex. 1.)

$$= -6\pi a^{\frac{1}{3}}\int^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}d(a^{\frac{2}{3}} - x^{\frac{2}{3}})$$

$$= \tfrac{12}{5}\pi a^2.$$

In this case an easier integral is obtained by expressing the surface in terms of y and dy, as in form (3). Thus,

$$\text{Surface} = 2 \cdot 2\pi \int_{y=0}^{y=a} y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = 4\pi a^{\frac{1}{3}} \int_0^a y^{\frac{2}{3}} dy = \tfrac{12}{5}\pi a^2.$$

2. Calculate the surface of the hypocycloid in Ex. 1, using the equations $x = a\cos^3\theta$, $y = a\sin^3\theta$.

3. Derive formula (5).

4. The cardioid $r = a(1 - \cos\theta)$ revolves about the initial line: find the area of the surface generated.

$$\text{Surface} = 2\pi \int_{\theta=0}^{\theta=\pi} PN \cdot ds.$$

Now $PN = r\sin\theta = a(1 - \cos\theta)\sin\theta$, and $ds = a\sqrt{2}\sqrt{1 - \cos\theta}\, d\theta$ (see Ex. 1, Art. 138).

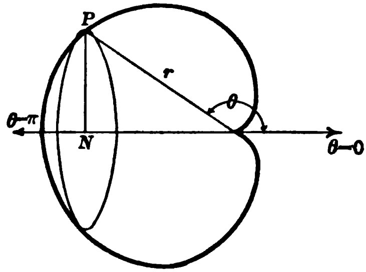

FIG. 76.

$$\therefore \text{ surface} = 2\sqrt{2}\,\pi a^2 \int_0^\pi (1 - \cos\theta)^{\frac{3}{2}} \sin\theta\, d\theta = \left[\frac{4\sqrt{2}}{5}\pi a^2 (1 - \cos\theta)^{\frac{5}{2}}\right]_0^\pi$$

$$= \tfrac{32}{5}\pi a^2.$$

5. Find the area of the spherical surface generated by the revolution of a circle of radius a about a diameter.

6. A quadrant of a circle of radius a revolves about the tangent at one extremity. What is the area of the curved surface generated?

7. Calculate the area of the surface of the prolate spheroid generated by the revolution of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x-axis.

8. In the case of an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, compute: (1) the area between the cycloid and the x-axis; (2) the volume and the surface generated by its revolution about the x-axis; (3) the volume and the surface generated by its revolution about the tangent at the vertex.

9. Find the volume and the surface generated by revolving the circle $x^2 + (y - b)^2 = a^2$, $(b > a)$, about the x-axis.

10. Find the area of the surface generated by the revolution of the arc of the catenary in Ex. 6, Art. 112.

11. The arc of the curve $r = a \sin 2\theta$, from $\theta = 0$ to $\theta = \frac{\pi}{4}$ (*i.e.* the first half of the loop in the first quadrant), revolves about the initial line: find the area of the surface generated. What is the area of the surface generated by the revolution of the second half of the same loop about the same line?

12. A circle is circumscribed about a square whose side is a. The smaller segment between the circle and one side of the square is revolved about the opposite side of the square. Find the volume and the surface of the solid ring thus generated.

140. Areas of surfaces whose equations have the form $z = f(x, y)$ or $F(x, y, z) = 0$. It is shown in solid geometry that:

(*a*) The cosine of the angle between the xy-plane and the tangent plane at any point (x, y, z) on such a surface, supposed to be continuous, is

$$\left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^{-\frac{1}{2}}. \tag{1}$$

(*b*) The area of the projection of a segment of a plane upon a second plane is obtained by multiplying the area of the segment by the cosine of the angle between the planes.

It follows from (*a*) and (*b*) that:

(*c*) If there be an area on the xy-plane equal to A, then A is the area that would be projected on the xy-plane by an area on the tangent plane at (x, y, z) which is equal to

$$A\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}. \tag{2}$$

(See C. Smith, *Solid Geometry*, Arts. 206, 26, 31; Murray, *Integral Calculus*, Art. 75.)

Let $z = f(x, y)$ be the equation of a surface $BFCRAGB$ [Fig. 66] whose area is required. Let $P(x, y, z)$ be any point on this surface, and P_1 the point $(x, y, 0)$ vertically below P. Let P_1Q_1 be a rectangle in the xy-plane having its sides equal to Δx and Δy respectively, and parallel to the x- and y-axes. Through the sides of this rectangle pass planes perpendicular to the xy-plane, and let these planes make with the surface the section PQ, and with the tangent plane at P the section PQ_2. (Q_1Q produced is supposed to meet in Q_2 the tangent plane at P.)

Then, $\qquad$ area $P_1Q_1 = \Delta x \cdot \Delta y.$

Hence, by (2), $\qquad$ area $PQ_2 = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \cdot \Delta y \cdot \Delta x.$

Now the smaller Δx and Δy become, the more nearly will the section PQ_2 on the tangent plane at P coincide with the section PQ on the surface. Accordingly, the more nearly will the sum of the areas of sections like PQ_2 on the tangent planes at points taken close together on the surface, become equal to the area of the surface; moreover, the difference between this sum and the area of the surface can be made as small as one pleases. Consequently, the area of the surface is the limit of the sum of the areas of these sections on the tangent planes when these sections become infinitesimal. That is,

$$\textbf{area } BFCRAGB = \int_{x=0}^{x=OA}\int_{y=0}^{y=SG}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\cdot dy\,dx.$$

NOTE. The integral $\left[\int_{y=0}^{y=SG}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dy\right]dx$ gives the area of the strip or zone RGL, and the integral $\int_{x=0}^{x=OA}RGLdx$ gives the sum of these zones from BOC to A.

EXAMPLES.

1. Find the area of the portion of the surface of the sphere in Ex. 7, Art. 132, that is intercepted by the cylinder.

The area required = 4 area $AVBLA$ (Fig. 68). In this figure, the equation of the sphere is

$$x^2+y^2+z^2=a^2,$$

and the equation of the cylinder is $x^2+y^2=ax$.

The area of a strip LV, two of whose sides are parallel to the zy-plane, will first be found; then the sum of all such strips in the spherical surface $AVBLA$ will be determined.

$$\text{Area } AVBLA = \int_{x=0}^{x=OA}\int_{y=0}^{y=RK}\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}}dy\,dx.$$

Since the required surface is on the sphere, the partial derivatives must be derived from the equation of the sphere.

Accordingly, $$\frac{\partial z}{\partial x}=-\frac{x}{z},\quad \frac{\partial z}{\partial y}=-\frac{y}{z};$$

hence, $$1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=1+\frac{x^2}{z^2}+\frac{y^2}{z^2}=\frac{a^2}{z^2}=\frac{a^2}{a^2-x^2-y^2}.$$

Also, $$RK=\sqrt{ax-x^2}.$$

$$\therefore \text{ area } AVBLA = \int_0^a\int_0^{\sqrt{ax-x^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}dy\,dx$$

$$= a\int_0^a\left[\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{ax-x^2}}dx$$

$$= a\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx.$$

This integral can be evaluated by integrating by parts. The integration can be simplified by means of the substitution $\sin z = \sqrt{\frac{x}{a+x}}$. It will be found that area required $= 4$ area $AVBLA = 2(\pi - 2)a^2 = 2.2832\, a^2$.

2. Find the area of the surface of the cylinder intercepted by the sphere in Ex. 7, Art. 132.

3. By the method of this article, find the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

4. A square hole is cut through a sphere of radius a, the axis of the hole coinciding with a diameter of the sphere: find the volume removed and the area of the surface cut out, the side of a cross-section of the hole being $2b$.

5. Find the area of that portion of the surface of the sphere intercepted by the cylinder in Ex. 4, Art. 133.

141. Mean values. In Art. 98 it has been stated that if the curve $y = f(x)$ be drawn (Fig. 44), and if $OA = a$ and $OB = b$, then, of all the ordinates from A to B,

$$\text{the mean value} = \frac{\text{area } APQB}{AB} = \frac{\int_a^b f(x)\,dx}{b-a}. \qquad (1)$$

Result (1) can be derived in the following way which has also the advantage of being adapted for leading up to a more general notion of mean value. The mean value of a set of quantities is defined as

$$\frac{\text{the sum of the values of the quantities}}{\text{the number of the quantities}}.$$

For instance, if a variable quantity takes the values 2, 5, 7, 9, the mean of these values is $\frac{2+5+7+9}{4}$ or $5\frac{3}{4}$.

Now take any variable, say x, and suppose that $f(x)$ is a continuous function, and let the interval from $x = a$ to $x = b$ be divided into n parts each equal to Δx, so that $n\,\Delta x = b - a$. Let the mean of the values of the function for the n successive values of x,

$$a,\ a + \Delta x,\ a + 2\,\Delta x,\ \cdots,\ a + \overline{n-1}\,\Delta x,$$

be required. The corresponding n successive values of the function are

$$f(a),\ f(a + \Delta x),\ f(a + 2\,\Delta x),\ \cdots,\ f(a + \overline{n-1}\cdot\Delta x).$$

Hence, mean value of function

$$=\frac{f(a)+f(a+\Delta x)+f(a+2\,\Delta x)+\cdots+f(a+\overline{n-1}\cdot\Delta x)}{n}. \qquad (2)$$

Now $n\,\Delta x = b - a$, whence $n = \dfrac{b-a}{\Delta x}$. Substitution in (2) gives mean value

$$=\frac{f(a)\Delta x+f(a+\Delta x)\Delta x+f(a+2\,\Delta x)\Delta x+\cdots+f(a+\overline{n-1}\,\Delta x)\Delta x}{b-a}. \qquad (3)$$

Finally, let the mean of all the values that $f(x)$ takes as x varies from a to b be required. In this case n becomes infinitely great and Δx becomes infinitesimal; accordingly [Art. 96 (2), (3)],

(3) becomes $$\textbf{mean value} = \frac{\int_a^b f(x)\,dx}{b-a}, \qquad (4)$$

as already represented geometrically in Art. 98.

NOTE 1. **Reference for collateral reading.** Echols, *Calculus*, Arts. 150–152.

EXAMPLES.

1. Find the mean length of the ordinates of a semicircle (radius a). the ordinates being erected at equidistant intervals on the diameter.

Choose the axes as in Fig. 77. Then the equation of the circle is $x^2 + y^2 = a^2$. Let PN denote any of the ordinates drawn as directed.

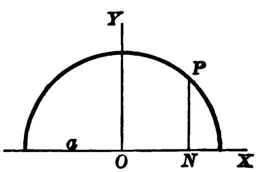

FIG. 77.

$$\text{Mean value} = \frac{\int_{-a}^{+a} PN\cdot dx}{a-(-a)} = \frac{\int_{-a}^{a}\sqrt{a^2-x^2}\,dx}{2\,a}$$

$$= \frac{\pi a^2}{2\cdot 2\,a} = .7854\,a.$$

2. Find the mean length of the ordinates of a semicircle (radius a), the ordinates being drawn at equidistant intervals on the arc.

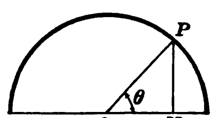

FIG. 78.

Let PN be any of the ordinates drawn at equidistant intervals on the arc, that is, at equal increments of the angle θ.

$$\text{Mean value} = \frac{\int_{\theta=0}^{\theta=\pi} PN\cdot d\theta}{\pi-0} = \frac{\int_0^{\pi} a\sin\theta\,d\theta}{\pi} = \frac{2\,a}{\pi} = .6366\,a.$$

NOTE 2. A slight inspection will show that it is reasonable to expect the results in Exs. 1, 2, to differ from each other.

SUGGESTION: Draw a number of ordinates, say 4 or 6 or 8, as specified in Ex. 1, and compare them with the ordinates of equal number drawn as specified in Ex. 2.

3. Find the average value of the following functions: (1) $7x^2+4x-3$ as x varies continuously from 2 to 6; (2) $x^3-3x^2+4x+11$ as x varies from -2 to 3. Draw graphs of these functions.

4. Find the average length of the ordinates to the parabola $y^2=8x$ erected at equidistant intervals from the vertex to the line $x=6$.

5. (1) In Fig. 51 find the mean length of the ordinates drawn from ON to the arc OML, and the mean length of the ordinates drawn from ON to the arc ORL. (2) In Fig. 50 find the mean length of the abscissas drawn from OY, (*a*) to the arc OR; (*b*) to the arc RL; (*c*) to the arc ORL. (3) In Fig. 52 find the mean ordinate from OL, (*a*) to the arc TKN; (*b*) to the arc TGM.

6. (1) In the ellipse whose semiaxes are 6 and 10, chords parallel to the minor axis are drawn at equidistant intervals: find their mean length. (2) In the ellipse in (1) find the mean length of the equidistant chords that are parallel to the major axis. (3) Do as in (1) and (2) for the general case in which the major and minor axes are respectively $2a$ and $2b$.

7. On the ellipse in Ex. 6, (3), successive points are taken whose eccentric angles differ by equal amounts: find the mean length of the perpendiculars from these points, (1) to the major axis; (2) to the minor axis.

8. In the case of a body falling vertically from rest, show that (1) the mean of the velocities at the ends of successive equal intervals of time, is one-half the final velocity; (2) the mean of the velocities at the ends of successive intervals of space, is two-thirds the final velocity. (The velocity at the end of t seconds is gt feet per second; the velocity after falling a distance s feet is $\sqrt{2gs}$ feet per second.)

9. A number n is divided at random into two parts: find the mean value of their product.

10. Find the mean distance of the points on a circle of radius a from a fixed point on the circle.

The interval $b-a$ in (1) and (4) through which the variable x passes is called *the range* of the variable, and dx is *an infinitesimal element of the range.* In (1) and Ex. 1 the range is a particular interval on the x-axis. In Ex. 2 the range is a certain angle, namely π; in Ex. 8 (2) the range is a vertical distance; in

Ex. 8 (1) the range is an interval of time. There are various other ranges at (or for) whose component parts a function may take different values. For instance, a curved line as in Ex. 10, a plane area as in Exs. 11, 13; a curved surface as in Ex. 15 (1); a solid as in Exs. 16, 17. The definition of mean value [or result (4)] may be extended to include such cases, thus:

$$\left.\begin{array}{l}\textbf{the mean value of a func-}\\ \textbf{tion over a certain range}\end{array}\right\} = \frac{\begin{array}{l}\textbf{lim } \Sigma\{(\textbf{value of function at each infini-}\\ \textbf{tesimal element of the range}) \times (\textbf{this}\\ \textbf{infinitesimal element})\}\end{array}}{\textbf{the range}}.$$

11. Find the mean square of the distance of a point within a square (side $= a$) from a corner of the square.

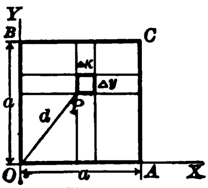

FIG. 79.

In this case "the range" extends over a square. Choose the axes as shown in Fig. 79. Take any point $P(x, y)$ in the range, and let its distance from O be d. At P let an infinitesimal element of the range be taken, viz. an element in the shape of a rectangle whose area is $dy\,dx$. Now $d^2 = x^2 + y^2$. $\therefore$ mean value of d^2 for all points in

$$OACB = \frac{\int_0^a \int_0^a (x^2+y^2)\,dy\,dx}{\text{area of square}} = \tfrac{2}{3}a^2.$$

12. Find (1) the mean distance, and (2) the mean square of the distance, of a fixed point on the circumference of a circle of radius a from all points within the circle. (Suggestion: use polar coördinates.)

13. Find (1) the mean distance, and (2) the mean square of the distance, of all the points within a circle of radius a from the centre.

14. Find the mean latitude of all places north of the equator.

15. For a closed hemispherical shell of radius a calculate (1) the mean distance of the points on the curved surface from the plane surface; (2) the mean distance of the points on the plane surface from the curved surface, distances being measured along lines perpendicular to the plane surface.

16. Calculate (1) the mean distance, and (2) the mean square of the distance, of all points within a sphere of radius a, from a fixed point on the surface.

17. Calculate (1) the mean distance, and (2) the mean square of the distance, of all points within a sphere of radius a, from the centre.

18. Find (1) the mean distance, and (2) the mean square of the distance, of all points on the surface of a sphere of radius a, from a fixed point on the surface.

19. Find (1) the mean distance, and (2) the mean square of the distance, of all points on a semi-undulation of the sine curve $y = a \sin x$, from the x-axis.

NOTE 3. The square root of the mean square in Ex. 19 (2) (viz. .7071 a) is of special importance in the measurement of alternating currents; for the heating and dynamometer effects of any current depend directly upon this square root. The latter is generally called "the mean square value of the ordinate of the sine curve" to distinguish it from "the average value" of this ordinate as found in Ex. 19 (1).

CHAPTER XVII.

CONCAVITY AND CONVEXITY. CONTACT AND CURVATURE. EVOLUTES AND INVOLUTES.

142. Concavity and convexity of curves: rectangular coördinates.

DEFINITION. At a point on a curve *the curve is said to be concave to a line* (*or to a point off the curve*) when an infinitesimal arc containing the point lies between the tangent at the point and the given line (or point off the curve). If the tangent lies between the line (or point) and the infinitesimal arc, the arc there is said to be *convex to the line* (*or point*).

Thus, in Fig. 20 *a*, at P the curve MN is concave to the line OX, and concave to the point A; in Fig. 20 *b*, at P_1 the curve MN is convex to the line OX, and convex to the point A. The arc on one side of a point of inflexion is concave to a given line (or point), and the arc on the other side of the point of inflexion is convex to this line (or point) (see Figs. 36 *a*, *b*).

The curves passing through P and R have the concavity towards the x-axis, and the curves passing through Q and S are convex to the x-axis. At P y is positive; and $\frac{d^2y}{dx^2}$ is negative, for $\frac{dy}{dx}$ decreases as a point moves along the curve towards the right through P. At R y is negative; and $\frac{d^2y}{dx^2}$ is positive, for $\frac{dy}{dx}$ increases as a point moves along the curve towards the right through R. Hence, *at points where a curve is concave to the x-axis* $y\frac{d^2y}{dx^2}$ *is negative.* A similar examination of the curves passing through Q and S shows that *at points where a curve is convex to the x-axis* $y\frac{d^2y}{dx^2}$ *is positive.*

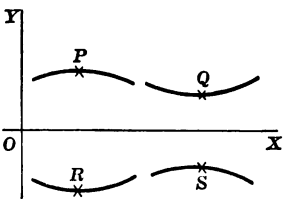

FIG. 80.

Ex. 1. Prove the theorem last stated.

Ex. 2. Test or verify the above theorems and Note 1 in the case of a number of the curves in the preceding chapters.

NOTE 1. The curves passing through P and S are *concave downwards*, and here $\frac{d^2y}{dx^2}$ *is negative*. The curves passing through R and Q are *concave upwards*, and here $\frac{d^2y}{dx^2}$ *is positive.*

NOTE 2. A point where a curve stops bending in one direction and begins to bend in the opposite direction as at L, A, D, H, G, P, Figs. 36 *a*, *b*, 37, is called *a point of inflexion.*

Note 3. A curve $f(r, \theta) = 0$ is concave or convex to the pole at the point (r, θ) according as $u + \frac{d^2u}{d\theta^2}$ is positive or negative, u denoting $\frac{1}{r}$. (See McMahon and Snyder, *Diff. Cal.*, Art. 144.)

143. Order of contact. If two curves, $y = \phi(x)$ and $y = f(x)$, intersect at a point at which $x = a$, as in Fig. 81 *a*, then $\phi(a) = f(a)$ and $\phi'(a) \neq f'(a)$. If $\phi(a) = f(a)$ and $\phi'(a) = f'(a)$, then the curves touch as in Fig. 81 *b*, and they are said to have *contact of the first order*, provided that $\phi''(a) \neq f''(a)$. If $\phi(a) = f(a)$, $\phi'(a) = f'(a)$, and $\phi''(a) = f''(a)$, but $\phi'''(a) \neq f'''(a)$, then the curves are said to

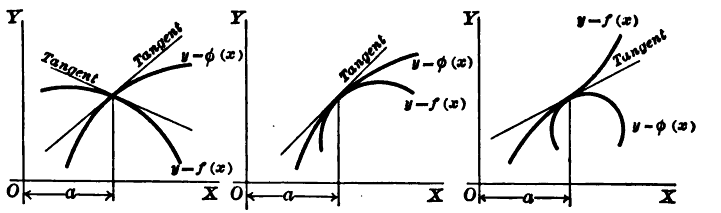

FIG. 81 *a*. FIG. 81 *b*. FIG. 81 *c*.

have *contact of the second order*, as in Fig. 81 *c*. And, in general, if $\phi(a) = f(a)$ and the respective successive derivatives of $\phi(x)$ and $f(x)$ up to and including the nth, but not including the $(n+1)$th, are equal for $x = a$, then *the curves are said to have contact of the nth order.* Hence, in order to find the order of contact of two curves compare the respective successive derivatives of y for the two curves at the points through which both curves pass.

NOTE 1. Another way of regarding contact is the following. In analytic geometry the tangent at P (Fig. 82 a) is defined as the limiting position which the secant PQ takes when PQ revolves about P until the point of intersection Q coincides with P. The line then has contact of the first order with the curve. This notion of points of intersection of a line and a curve becoming coincident will now be extended to curves in general. Two curves,

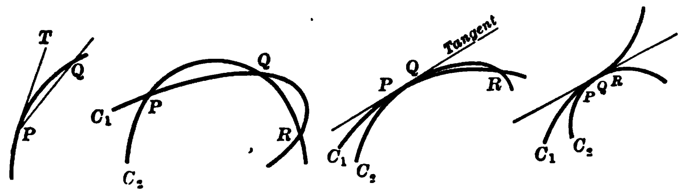

FIG. 82 a. FIG. 82 b. FIG. 82 c. FIG. 82 d.

C_1 and C_2 (Fig. 82 b), are said to intersect when they have a point, as P, in common. They are said to have contact of the *first order* at P when the curves (see Fig. 82 c) have been modified in such a way that a *second point* of intersection Q moves into coincidence with P. (The value of $\frac{dy}{dx}$ at P is then the same for both curves, according to the definition of a tangent as given above.) The curves are said to have contact of the *second order* at P when the curves have been further modified in such a way that a *third point* of intersection R moves into coincidence with P and Q (see Fig. 82 d). (The value of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, *i.e.* $\frac{d^2y}{dx^2}$, is then the same for both curves at P.) And, in general, the curves are said to have *contact of the nth order* at a point P when $n+1$ of their points of intersection have moved into coincidence with P. (At P the respective derivatives of y up to the nth are then the same for both curves.) See Echols, *Calculus*, Art. 98.

NOTE 2. *In general a straight line cannot have contact of an order higher than the first with a curve.* For in order that a line have contact of the first order with a curve at a given point, the ordinates of the line and the curve must be equal there, and likewise their slopes; thus two equations must be satisfied. These equations suffice to determine the two arbitrary constants appearing in the equation of a straight line. For example, if the line $y = mx + b$ has contact of the first order with the curve $y = f(x)$ at the point for which $x = a$, the following two equations are satisfied, viz.:

$$f(a) = ma + b, \ f'(a) = m;$$

from these equations m and b can be found.

This line and curve have contact of the second order in the particular (and exceptional) case in which $f''(a) = 0$; consequently (Art. 78), if there is a

point of inflexion on the curve $y = f(x)$ where $x = a$, the tangent there has contact of the second order.

The theorem at the beginning of this note is also evident from geometrical considerations. Since, in general, a line can be passed through only two arbitrarily chosen points of a curve, it is to be expected from Note 1 that in general a line and a curve can have contact of the first order only.

NOTE 3. *In general, a circle cannot have contact of an order higher than the second with a curve.* For in order that a circle have contact of the second order with a curve at a given point, three equations must be satisfied, and these equations just suffice to determine the three arbitrary constants that appear in the general equation of a circle [see Eq. (2), Art. 144]. This theorem is also evident from Note 1 and the fact that, in general, a circle can be passed through only three arbitrarily chosen points of a curve. (In a few very special instances a circle has contact of the third order with a curve. See Ex. 4, Art. 149.)

NOTE 4. It is shown in Art. 182 that *when two curves have contact of an odd order, they do not cross each other at the point of contact;* but *when they have contact of an even order, they do cross there. Illustrations:* the tangent at an ordinary point on a curve, as shown in Figs. 15, 17; the tangent at a point of inflexion, as in Figs. 31 *a*, *b*, 36, 37; an ellipse and circles having contact of second order therewith (see Ex. 4, Art. 149). This theorem may also be deduced from geometry and the definitions given in Note 1.

N.B. As far as possible make good figures showing the curves, lines, and points mentioned in the exercises in this chapter.

EXAMPLES.

1. Find the place and order of contact of (1) the curves $y = x^3$ and $y = 6x^2 - 9x + 4$; (2) the curves $y = x^3$ and $y = 6x^2 - 12x + 8$.

2. Determine the parabola which has its axis parallel to the y-axis, passes through the point (0, 3), and has contact of the first order with the parabola $y = 2x^2$ at the point (1, 2).

3. What must be the value of a in order that the parabola $y = x + 1 + a(x-1)^2$ may have contact of the second order with the hyperbola $xy = 3x - 1$?

4. Find the parabola whose axis is parallel to the y-axis, and which has contact of the second order with the cubical parabola $y = x^3$ at the point (1, 1).

5. Determine the parabola which has its axis parallel to the y-axis and has contact of the second order with the hyperbola $xy = 1$ at the point (1, 1).

144. Osculating circle. It was pointed out in Art. 143, Note 3, that contact of the second order is, in general, the closest contact that a circle can have with a curve. A circle having contact of the second order with a curve at a point is called *the osculating circle* at that point.

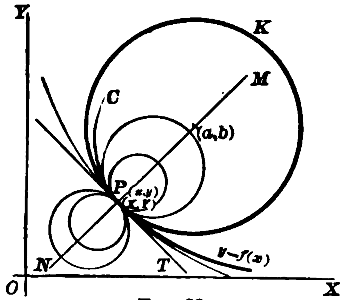

FIG. 83.

In Fig. 83 PT is tangent to the curve C at P. Every circle which passes through P and has its centre in the normal NM touches C at P. One of these circles has contact of the second order with C at P; let this circle be denoted by K. All the other circles, infinite in number, in general have contact of the first order only.

Osculating circle: rectangular coördinates. The radius and the centre of the osculating circle at any point $P(x, y)$ on the curve

$$y = f(x) \tag{1}$$

will now be obtained. Denote the centre and radius by (a, b) and r. Then the equation of the osculating circle at the point (x, y) is

$$(X - a)^2 + (Y - b)^2 = r^2. \tag{2}$$

For the moment, for the sake of distinction, x and y are used to denote the coördinates of a point on the curve, and X and Y are used to denote the coördinates of a point on the circle. Then at the point where the circle and the curve have contact of the second order

$$X = x,\ Y = y,\ \frac{dY}{dX} = \frac{dy}{dx},\ \frac{d^2Y}{dX^2} = \frac{d^2y}{dx^2}. \tag{3}$$

From (2), on differentiating twice in succession,

$$X - a + (Y - b)\frac{dY}{dX} = 0, \tag{4}$$

$$1 + \left(\frac{dY}{dX}\right)^2 + (Y - b)\frac{d^2Y}{dX^2} = 0. \tag{5}$$

$$\therefore\ Y - b = -\left[1 + \left(\frac{dY}{dX}\right)^2\right] \div \frac{d^2Y}{dX^2}, \tag{6}$$

and
$$X - a = \left[1 + \left(\frac{dY}{dX}\right)^2\right]\frac{dY}{dX} \div \frac{d^2Y}{dX^2}. \tag{7}$$

Accordingly, from (3), (2), (6), (7),

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}; \tag{8}$$

and from (3), (6), (7),

$$a = x - \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx};\ b = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{9}$$

NOTE. For the osculating circle, polar coördinates being used, see Art. 150, Note 2.

Ex. 1. Determine the radius and the centre of the osculating circle for each of the curves in Ex. 1 (1), Art. 143, at their point of contact.

Ex. 2. Do as in Ex. 1 for the curves Ex. 1 (2), Art. 143.

145. The notion of curvature. Let the curves A, B, C, D have a common tangent PT at P. At the point P the curve A, to use the popular phrase, bends or curves more than the curves B, C, and D; and D bends or curves less than the curves A, B, and C. These four curves evidently differ in the rate at which they bend, or turn away from the straight line PT, at P. These ideas are sometimes expressed by saying that these curves differ in curvature at P, and that there A has the greatest and D the least curvature. In the case of two circles, say one with a radius of an inch and the other with a radius of a million miles, it is customary to say that the second circle has a small curvature, and that the first has a large curvature in comparison with the second. An inspection of a figure consisting of a circle and some of its tangents gives the impression that what is popularly called the curvature is the same at all points of that circle.

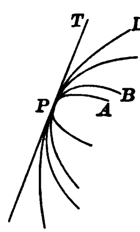

FIG. 84.

On the other hand, an inspection of an elongated ellipse gives the impression that the curvature is not the same at all points of that ellipse, although at two particular points, or at four particular points, it may be the same. Curvature will now be given a precise mathematical definition and its measurement will be explained.

Ex. 1. Draw an ellipse, and find by inspection the points where the curvature is greatest and where it is least. Show how to obtain sets of four points on the ellipse which have the same curvature.

Ex. 2. Discuss a parabola and an hyperbola in the manner of Ex. 1.

146. Total curvature. Average curvature. Curvature at a point. At A_1 the curve C has the direction A_1T_1, which makes the angle ϕ_1 with the x-axis; at A_2 the curve has the direction A_2T_2, which makes an angle ϕ_2 with the x-axis. The difference between these directions represents the angle by which the curve has changed its direction from the direction of the line A_1T_1 in the interval of arc from A_1 to A_2. This difference, namely, T_1RT_2 or $\phi_2 - \phi_1$, is called the *total curvature of the arc* A_1A_2.

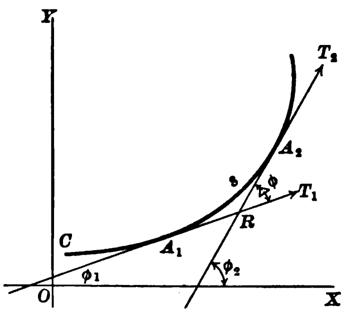

FIG. 85.

The *average curvature for this arc* is

$$(\phi_2 - \phi_1) \div \text{length of arc } A_1A_2.$$

(Here the angle is measured in *radians.*)

Accordingly, if (Fig. 86) $\Delta\phi$ is the angle between the tangents at A and B, then $\Delta\phi$ is the total curvature of the arc AB; if Δs is the length of the arc AB, then $\dfrac{\Delta\phi}{\Delta s}$ is the average curvature of that arc. Now let B approach A. The arc Δs and the angle $\Delta\phi$ then become infinitesimal; and, finally, when B reaches A, $\dfrac{\Delta\phi}{\Delta s}$ has the

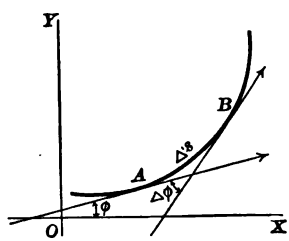

FIG. 86.

limiting value $\frac{d\phi}{ds}$. The $\text{limit}_{\Delta s \doteq 0} \frac{\Delta\phi}{\Delta s}$ at any point on a curve, *i.e.* $\frac{d\phi}{ds}$ there, is called **the curvature of the curve** *at that point.* (The phrase "curvature of a curve" means the curvature of the curve *at a particular point.*) In all curves, with the exception of straight lines and circles, the curvature, in general, varies from point to point.

147. The curvature of a circle. Let A and B be two points on a circle having its centre at O. In Fig. 87 the angle between the directions of the tangents AT_1 and BT_2 is $\Delta\phi$, say. Let Δs denote the length of the arc AB. Then $AOB = T_1RT_2 = \Delta\phi$. Hence, by trigonometry, $\Delta s = r\Delta\phi$.

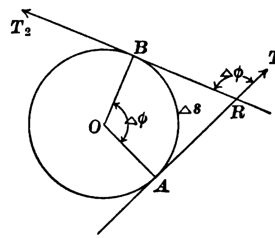

FIG. 87.

From this,

$$\frac{\Delta\phi}{\Delta s} = \frac{1}{r}; \text{ whence } \frac{d\phi}{ds} = \frac{1}{r}. \qquad (1)$$

That is, the curvature of a circle is constant and is the reciprocal of (*the measure of*) the radius.

NOTE. When the radius increases beyond all bounds, the curvature approaches zero, and the circle approaches a straight line as its limiting position. When the radius decreases, the curvature increases; as the radius approaches zero and the circle thus shrinks towards a point, the curvature approaches an infinitely great value.

It is shown in Ex. 5, Art. 194, that all curves of constant curvature are circles.

Ex. Compare the curvatures of circles of radii 2 inches, 2 feet, 5 yards, 2 miles, 10 miles, 100 miles, and 1,000,000 miles.

148. To find the curvature at any point of a curve: rectangular coördinates. Let the curve in Fig. 86 be $y = f(x)$, and let its curvature at any point $A(x, y)$ be required. Let k denote the curvature at A, and ϕ denote the angle which the tangent at A makes with the x-axis. Take an arc AB and denote its length by Δs, and denote the angle between the tangents at A and B by $\Delta\phi$. Then, by the definition in Art. 146,

$$k = \frac{d\phi}{ds} \text{ at } A.$$

Now (Art. 58), $\tan\phi = \frac{dy}{dx}$. $\therefore\ \phi = \tan^{-1}\frac{dy}{dx}$.

$$\therefore\ k = \frac{d\phi}{ds} = \frac{d}{ds}\left(\tan^{-1}\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan^{-1}\frac{dy}{dx}\right)\cdot\frac{dx}{ds} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} \div \frac{ds}{dx}.$$

$$\therefore\ [\text{Art. 67 } c(2)],\qquad k = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{1}$$

This, by (1) Art. 147 and (8) Art. 144, is the same as the curvature of the osculating circle.

In order to find the curvature at a definite point (x_1, y_1) it is only necessary to substitute the coördinates x_1, y_1, in the general result (1).

Ex. 1. Compute and compare the curvatures of the two curves in Ex. 1 (1), Art. 143, at their point of contact.

Ex. 2. Find the curvature of the curve $y = x^3 - 2x^2 + 7x$ at the origin. Determine the radius and centre of its osculating circle at that point.

149. The circle of curvature at any point on a curve: rectangular coördinates. *The circle of curvature at a point on a curve* is the circle which passes through the point and has the same tangent and the same curvature as the curve has there. The radius of this circle is called *the radius of curvature* at the point, and the centre of the circle is called *the centre of curvature* for the point.

The radius of curvature. Let R denote the radius of curvature and (α, β) denote the centre of curvature for any point (x, y) on the curve $y = f(x)$. Then it follows from Art. 147, and Art. 148, Eq. 1, that

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \tag{1}$$

(That is, R is the value of this expression at that point.)

NOTE 1. There is an infinite number of circles that can pass through a given point on a curve and have the same tangent as the curve has there but not the same curvature, and there is an infinite number of circles that can

pass through this point and have the same curvature but not the same tangent as the curve has there; but there is *only one* circle passing through the point that has there both the same tangent and the same curvature as the curve.

Ex. 1. Illustrate Note 1 by figures.

The centre of curvature. Since at any point on a curve the circle of curvature and the curve have the same tangent and curvature, it follows that $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are respectively the same for the circle and the curve at that point. Accordingly (Art. 143, Note 3) the circle of curvature has, in general,* contact of the second order with the curve, and thus (Art. 144) coincides with the osculating circle passing through the point. Accordingly (Art. 144, Eq. 9)

$$\alpha = x - \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx}; \quad \beta = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{2}$$

NOTE 2. The coördinates of the centre of curvature may also be obtained in the following manner.

Let C be the centre of the circle of curvature of the curve PL at P, and let the tangent PT make the angle ϕ with the x-axis. Draw the ordinates PM and CN, and draw PB parallel to OX. Let R denote the radius of curvature. Then $NCP = \phi$, and $\tan\phi = \frac{dy}{dx}$.

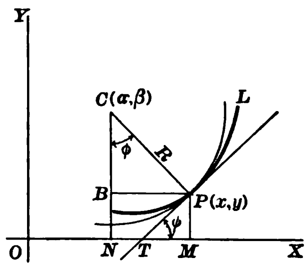

FIG. 88.

In Fig. 88

$$\alpha = ON = OM - BP = x - R\sin\phi$$

$$= x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{\frac{dy}{dx}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}} = x - \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx}. \tag{3}$$

Also, $\beta = NC = MP + BC = y + R\cos\phi = y + \dfrac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$ (4)

The results for Fig. 88 are true for all figures.

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* For an exception see the circles of curvature at the ends of the axes of an ellipse. (See Ex. 4 following.)

Ex. 2. Verify the last statement by drawing the radii of curvature at points on each side of points of maximum and minimum in the curves in Fig. 80 and carefully noting the algebraic signs of $\frac{dy}{dx}$ and $\frac{d^2y}{d^2x}$ at these points.

NOTE 3. A glance at Fig. 80 shows that at P and R the normal (Art. 59) and the radius of curvature have the same direction, and at Q and S they have opposite directions. Hence (see Art. 142) the normal and the radius of curvature at a point on a curve have the same or opposite directions according as $y\frac{d^2y}{dx^2}$ there is respectively negative or positive.

NOTE 4. At a point of inflexion, according to Art. 78, and Art. 148, Eq. (1), the curvature is zero.

NOTE 5. *A centre of curvature is the limiting position of the intersection of two infinitely near normals to the curve.* For a consideration of this important geometrical fact, see Williamson, *Diff. Cal.* (7th ed.), Art. 229; Lamb, *Calculus*, Art. 150; Gibson, *Calculus*, Art. 141.

EXAMPLES.

3. Find the radius of curvature and the centre of curvature at any point on the parabola $y^2 = 4px$. What are they for the vertex?

Apply the general results just obtained to particular cases, by giving p particular values, *e.g.* 1, 2, etc., and taking particular points on the curves, and make the corresponding figures.

N.B. As in Ex. 3, apply the general results obtained in the following examples to particular cases.

4. As in Ex. 3 for the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Find the radii of curvature at the ends of the axes. Show that this radius at an extremity of the major axis is equal to half the latus rectum. Illustrate Note 4, Art. 143, by drawing an ellipse and the circles of curvature at various points on it. Show that the circles of curvature for an ellipse, at the ends of the axes, have contact of the third order with the ellipse.

5. Find the radius and centre of curvature at any point of each of the following curves: (1) The hyperbola $b^2x^2 - a^2y^2 = a^2b^2$. (2) The hyperbola $xy = a^2$. (3) The catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$. (4) The astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (5) The astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. (6) The semi-cubical parabola $x^3 = ay^2$. (7) The curve $x^2y = a^2(x - y)$ where $x = a$. (8) The cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. In this cycloid show that the length of the radius of curvature at any point is twice the length of the normal.

6. Find the radius of curvature at any point of each of the following curves: (1) The parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$. In this curve show that $\alpha + \beta = 3(x + y)$. (2) The cubical parabola $a^2y = x^3$. (3) The catenary of uniform

strength $y = c \log \sec\left(\frac{x}{c}\right)$. (4) The witch $xy^2 = a^2(a - x)$ at the vertex. (5) The parabola $x = a \cot^2 \psi$, $y = 2a \cot \psi$. (6) The ellipse $x = a \cos \phi$, $y = b \sin \phi$. (7) The hyperbola $x = a \sec \phi$, $y = b \tan \phi$. (8) The catenary $x = a \log(\sec \theta + \tan \theta)$, $y = a \sec \theta$.

150. The radius of curvature: polar coördinates. This can be deduced (*a*) directly from the definition of curvature (Art. 146) and the definition of radius of curvature (Art. 149); and (*b*) from form (1), Art. 149, by the usual substitution for transformation of coördinates, namely, $x = r\cos\theta$, $y = r \sin \theta$.

(*a*) By Art. 60 (2), $\phi = \theta + \psi$.

Now $$k = \frac{d\phi}{ds} \text{ (Art. 146) } = \frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds} = \left(1 + \frac{d\psi}{d\theta}\right)\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{-\frac{1}{2}}.$$

[Art. 67 *d*, Eq. (3).]

Also, $\tan\psi = r\frac{d\theta}{dr}$ (Art. 60). $\therefore \psi = \tan^{-1}\left[\frac{r}{\frac{dr}{d\theta}}\right]$. $\therefore \frac{d\psi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}$.

Hence $$R = \frac{1}{k} = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}. \tag{1}$$

(*b*) The deduction of (1) from (1), Art. 149, by the transformation of coördinates is left as an exercise for the student.

NOTE 1. On the substitution of u for $\frac{1}{r}$ in (1), $R = \frac{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{\frac{3}{2}}}{u^3\left(u + \frac{d^2u}{d\theta^2}\right)}$.

NOTE 2. Since the osculating circle and the circle of curvature coincide, the forms just found for R give *the radius of the osculating circle.*

NOTE 3. For other expressions for R see Todhunter, *Diff. Cal.*, Art. 321, and Ex. 4, page 352; Williamson, *Diff. Cal.* (7th ed.), Art. 236. Also see F. G. Taylor, *Calculus*, Arts. 288–290.

EXAMPLES.

1. Find the radius of curvature at any point of each of the following curves: (1) The circles $r = a$ and $r = 2b\cos\theta$. (2) The parabola $r(1 + \cos\theta) = 2a$. (3) The cardioid $r = a(1 + \cos\theta)$. (4) The equilateral hyperbola $r^2 \cos 2\theta = a^2$. (5) The lemniscate $r^2 = a^2 \cos 2\theta$. (6) The logarithmic spiral $r = e^{a\theta}$. (7) The spiral of Archimedes $r = a\phi$. (8) The general spiral $r = a\phi^n$.

2. Derive the expression for R in Note 1.

151. Evolute of a curve. Corresponding to each point on a given curve there is a centre of curvature. The locus of the centres of curvature for all the points on the curve, is called *the evolute of the curve.* Thus, if AA_1 be the given curve and C_1, C_2, C_3, $\cdots$, be respectively the centres of curvature for any points A_1, A_2, A_3, $\cdots$, on the given curve, the curve $C_1C_2C_3$ is the evolute of AA_1.

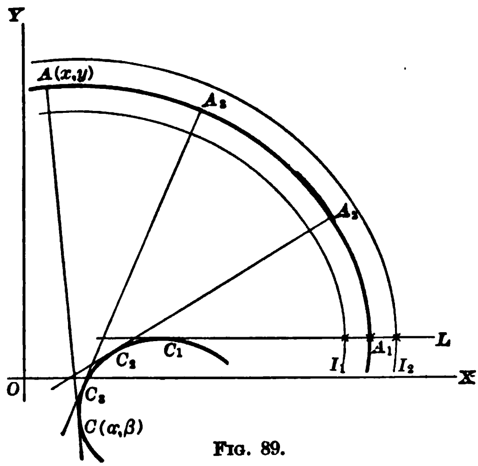

FIG. 89.

To find the equation of the evolute of the curve. Let the equation of the given curve be

$$y = f(x), \qquad (1)$$

and let $A(x, y)$ be any point on it. Let C be the centre of curvature for the point A, and denote C by (α, β). Then [Art. 149, Eq. (2)],

$$x - \alpha = \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx}, \qquad (2)$$

$$y - \beta = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \qquad (3)$$

On the elimination of x and y from equations (1), (2), (3), there will appear an equation which is satisfied by α and β, the coördinates of the point C. But A is any point on the given curve, and, accordingly, C is any of the centres of curvature for the points on AA_1. Accordingly, the equation found as indicated is the equation of the evolute.

NOTE. The algebraic process of eliminating x and y from (1), (2), and (3) depends on the form of these equations.

EXAMPLES.

1. Find the evolute of the parabola

$$y^2 = 4px. \tag{1}$$

Here by Ex. 3, Art. 149,
$$\alpha = 2p + 3x; \tag{2}$$
$$\beta = \frac{y^3}{4p^2}. \tag{3}$$

The elimination of x and y between equations (1), (2), (3), gives the equation of the evolute, viz. the semi-cubical parabola

$$4(\alpha - 2p)^3 = 27p\beta^2;$$

i.e. on using the ordinary notation for the coördinates,

$$4(x - 2p)^3 = 27py^2.$$

2. Find the evolute of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. (1)

Here, by Ex. 4, Art. 149,
$$\alpha = \left(\frac{a^2 - b^2}{a^4}\right)x^3, \tag{2}$$
$$-\beta = \left(\frac{a^2 - b^2}{b^4}\right)y^3. \tag{3}$$

The elimination of x and y between equations (1), (2), (3), gives the equation of the evolute, viz.:

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

i.e. on using the ordinary notation for coördinates,

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

3. Find the evolute of the following curves: (1) the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$. (2) The equilateral hyperbola $xy = a^2$. (3) The four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

4. Find both geometrically and analytically the evolute of a circle.

5. Show that the evolute of a complete arch of a cycloid consists of the halves of an equal cycloid. [Suggestion: see Ex. 5 (8), Art. 149.]

152. Properties of the evolute. The two most important properties of the evolute of a curve are the following:

(*a*) *The normal at any point of a given curve is a tangent to the evolute, and any tangent to the evolute is a normal to the given curve.*

(*b*) *The length of an arc of an evolute,* provided that the curvature varies continuously from point to point along this arc, *is equal to the difference between the lengths of the two radii of curvature drawn from the given curve to the extremities of the arc.*

Proof of (a). Let AA_1 (Fig. 89) be the given curve, and let its equation be $y = f(x)$, and let CC_1 be its evolute. Let $C(\alpha, \beta)$ be the centre of curvature for any point $A(x, y)$.

The slope of the given curve at A is $\frac{dy}{dx}$, and the slope of the evolute at C is $\frac{d\beta}{d\alpha}$. From Equations (2), Art. 149, on differentiation and reduction,

$$\frac{d\beta}{dx} = \frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}}{\left(\frac{d^2y}{dx^2}\right)^2}, \tag{1}$$

$$\frac{d\alpha}{dx} = \frac{-\frac{dy}{dx}\left\{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}\right\}}{\left(\frac{d^2y}{dx^2}\right)^2}. \tag{2}$$

From (1) and (2), and Art. 34 (3),

$$\frac{d\beta}{d\alpha} = \left(\frac{d\beta}{dx} \div \frac{d\alpha}{dx}\right) = -\frac{dx}{dy}. \tag{3}$$

But $-\frac{dx}{dy}$ is the slope of the normal at $A(x, y)$. Hence, the normal at A and the tangent to the evolute at C coincide.

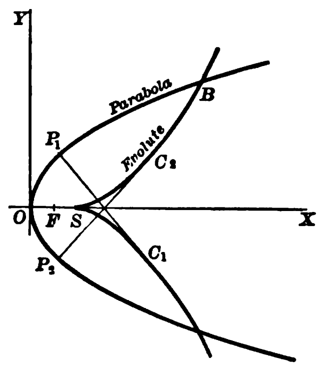

FIG. 90 a.

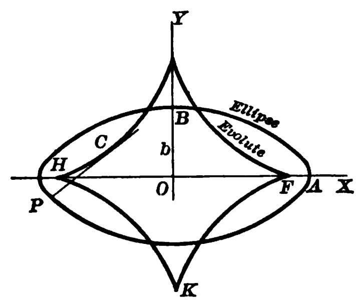

FIG. 90 b.

NOTE 1. Thus, in Fig. 89, AC is the radius of curvature for A on AA_1, AC is normal to AA_1 at A, and AC touches the evolute CC_1 at C. In Figs. 90 a, 90 b, P_1C_1, P_2C_2, are normal to the parabola and tangent to its evolute; PC is normal to the ellipse and tangent to its evolute.

NOTE 2. On account of property (*a*) the evolute is sometimes defined as the envelope (see Art. 154) of the normals of the curve. See Art. 157 (Ex. 2 and Notes 4, 5) and Art. 158, Ex. 1. Also see Echols, *Calculus*, Arts. 106–108.

Proof of (*b*). Let K be the given curve $y = f(x)$, and E its evolute.

Let C_1 be the centre of curvature for A_1, and C_2 the centre of curvature for A_2. Denote any point in K by (x, y), the radius of curvature there by R, and the corresponding centre of curvature in E by (α, β). Let the points A_1, A_2, C_1, C_2, be denoted as (x_1, y_1), (x_2, y_2), (α_1, β_1), (α_2, β_2), respectively; also let the radii of curvature A_1C_1 and A_2C_2 be denoted by R_1 and R_2. It will now be shown that

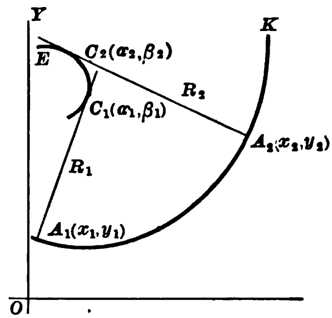

FIG. 91.

length of arc $C_1C_2 = R_2 - R_1$.

$$\text{Arc } C_1C_2 = \int_{\beta=\beta_1}^{\beta=\beta_2} \sqrt{1+\left(\frac{d\alpha}{d\beta}\right)^2} \cdot d\beta. \quad \text{(See Art. 137.)} \qquad (4)$$

On substituting the value of $\frac{d\alpha}{d\beta}$ from (3), and the value of $d\beta$ derived from (1), and noting that

$$x = x_1 \text{ when } \beta = \beta_1, \text{ and } x = x_2 \text{ when } \beta = \beta_2,$$

Equation (4) becomes

$$\text{arc } C_1C_2 = \int_{x=x_1}^{x=x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \left\{ \frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 - \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}}{\left(\frac{d^2y}{dx^2}\right)^2} \right\} dx. \quad (5)$$

Differentiation of R in Art. 149, Eq. (1), will show that $\frac{dR}{dx}$ is the same as the integrand in (5). Then, since $R = R_1$ when $x = x_1$, and $R = R_2$ when $x = x_2$, and $\frac{dR}{dx}dx = dR$ (Art. 27), Equation (5) becomes

$$\text{arc } C_1C_2 = \int_{x=x_1}^{x=x_2} \frac{dR}{dx} dx = \int_{x=x_1}^{x=x_2} dR = \int_{R=R_1}^{R=R_2} dR = R_2 - R_1.$$

Note 3. See Echols, *Calculus*, Art. 170 and Chap. XIV.

Ex. 1. Show that the total length of the evolute of the ellipse whose semi-axes are a and b, is $\frac{4(a^3 - b^3)}{ab}$.

Ex. 2. Show that the length of the evolute of the parabola $y^2 = 4px$ that is intercepted by the parabola (*i.e.* $2\,SB$, Fig. 90 a) is $4p(3\sqrt{3} - 1)$.

153. Involutes of a curve. In Fig. 89 the curve CC_1 is the evolute of the curve AA_1. Suppose that a string is stretched tightly along the curve CC_1 and held taut in the position $LC_1C_2C_3C$, the portion LC_1 thus being tangent to the evolute at C_1. Now, a point A_1 being taken in the string, let it be unwound from C_1C. It follows from properties (a) and (b), Art. 152, that, as the string is unwound from the evolute C_1C, A_1 will describe the curve A_1A. It is on account of this property that CC_1 is called the evolute of AA_1. On the other hand, AA_1 is called *an involute* of CC_1. "*An* involute," because CC_1 has an infinitely great number of involutes. For, when the string is unwound from the evolute C_1C an involute will be traced out by *each* point like A_1 taken in the string $LA_1C_1C_2C_3$. These involutes are *parallel curves* *; for (1) they have the same normals, namely, the tangents of their common evolute, and (2) the distance between any two of them along these normals is constant, namely, the distance between the two points originally taken on the string that is being unwound. Figure 89 shows three involutes of CC_1.

EXAMPLES.

1. Construct several involutes of the evolute of the parabola whose latus rectum is 8 (besides the parabola itself).

2. Construct several involutes of the evolute of the ellipse whose axes are 9 and 25.

3. Given a cycloid, construct the involute that is traced out by the point at the vertex in the course of "the unwinding."

4. Given a circle, construct the involute that is traced out by any point on the circle in the course of "the unwinding." (In the case of a circle all such involutes are identically equal. Accordingly, such an involute is usually termed "*the* involute of the circle.")

5. Construct several involutes of an ellipse, and several involutes of a parabola.

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* Two curves are said to be *parallel* when they have common normals always differing in length by the same amount.

CHAPTER XVIII.

SPECIAL TOPICS RELATING TO CURVES.

ENVELOPES, ASYMPTOTES, SINGULAR POINTS, CURVE TRACING.

ENVELOPES.

154. Family of curves. Envelope of a family of curves. The idea of a family of curves may be introduced by an example. The equation

$$(x-c)^2+y^2=4 \qquad (1)$$

is the equation of a circle of radius 2 whose centre is at $(c, 0)$. If c be given particular values, say 2, 3, -5, the equations of particular circles are obtained. Thus Equation (1) really represents a *family* of circles, viz. the circles (see Fig. 92) whose radii

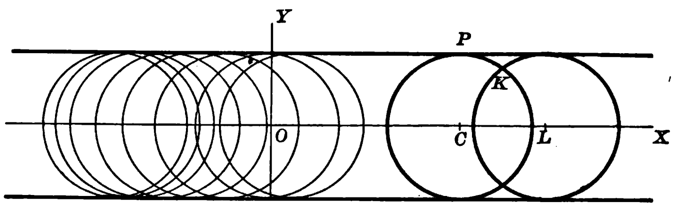

FIG. 92.

are 2 and whose centres are on the x-axis. The individual members of the family are obtained by letting c change its values from $-\infty$ to $+\infty$. A number such as c, whose different values serve to distinguish the individual members of a family of curves, is called *the parameter of the family*. Thus, to take another example, the equation $y=2x+b$ represents the family of straight lines having the slope 2; and $y=2x+5$, $y=2x-7$, are particular lines of the family. (Let a figure be constructed.) In this case the parameter b can take all values from $-\infty$ to $+\infty$.

To generalize: $$f(x, y, a) = 0 \tag{2}$$
is the equation of a family of curves whose parameter is a. The individual members or curves of the family are obtained by giving particular values to a. These curves are all of the same kind, but differ in various ways; for instance, in position, shape, or enclosed area. A family of curves may have two or more parameters. Thus, $y = mx + b$, in which m and b may take any values, has two parameters m and b, and represents all lines. The equation $(x-h)^2 + (y-k)^2 = 25$, in which h and k may take any values, represents all circles of radius 5. The equation $(x-h)^2 + (y-k)^2 = r^2$, in which h, k, and r may each take any value, represents all circles.

Envelope. *The envelope of a family of curves* is the curve, or consists of the *set* of curves, which touches every member of the family and which, at each point, is touched by some member of the family. For example, the envelope of the family of circles in Fig. 92 evidently consists of the two lines $y-2=0$ and $y+2=0$. On the other hand, the family of parallel straight lines $y=2x+b$ does not have an envelope; and, obviously, a family of concentric circles cannot have an envelope.

EXAMPLES.

1. Say what family of curves is represented by each of the following equations, and in each instance make a sketch showing several members of the family:

(*a*) $x^2 + y^2 = r^2$, parameter r. (*b*) $y = mx + 4$, parameter m.

(*c*) $y^2 = 4px$, parameter p. (*d*) $y^2 = 4a(x + a)$, parameter a.

(*e*) $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$, parameter a. (*f*) $\frac{x^2}{16+k} + \frac{y^2}{9^2+k} = 1$, parameter k.

(*g*) $y = mx + \frac{2}{m}$, parameter m. (*h*) $y = mx + \sqrt{25m^2 + 16}$, parameter m.

2. Express opinions as to which of the families in Ex. 1 have envelopes, and as to what these envelopes may be.

155. Locus of the ultimate intersections of the curves of a family. In Eq. (2), Art. 154, the equation of a family of curves, let a be given the particular value a_1; then there is obtained the equation of a particular member of that family, viz.

$$f(x, y, a_1) = 0, \tag{1}$$

Also, $$f(x, y, a_1 + h) = 0$$

is the equation of another member of the family. Let I. and II. be these curves. The smaller h becomes, the more nearly does curve II. come into coincidence with curve I. Moreover, as h becomes smaller and approaches zero, A, the point of intersection of these curves, approaches a definite limiting position. For example, if (Fig. 92) the centre L approaches nearer to C, then K, the point of intersection of the circles whose centres are at C and L, moves nearer to P; and finally, when L reaches C, K arrives at the definite position P. The locus of the limiting position of the point (or points) of intersection of two curves of a family which are approaching coincidence is called *the locus of ultimate intersections of the curves of the family.* For instance, in the case of the family of circles in Fig. 92, this locus evidently consists of the lines $y - 2 = 0$ and $y + 2 = 0$.

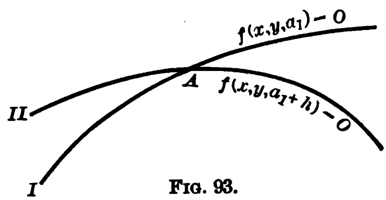

FIG. 93.

NOTE. The last-mentioned locus may also be derived analytically.

Let $$(x - c_1)^2 + y^2 = 4 \tag{1}$$

and $$(x - c_1 - h)^2 + y^2 = 4 \tag{2}$$

be two of the circles. On solving these equations simultaneously in order to find the point of intersection, there is obtained

$$(x - c_1)^2 - (x - c_1 - h)^2 = 0; \text{ whence } h(2x - 2c_1 - h) = 0,$$

and, accordingly, $$x = c_1 + \frac{h}{2}.$$

An ultimate point of intersection is obtained by letting h approach zero. If $h \doteq 0$, then $x \doteq c_1$, and by (1) $y \doteq \pm 2$. Thus $y = \pm 2$ at the ultimate points of intersection, and therefore the locus of these points is the pair of lines $y = \pm 2$.

N.B. In the following articles "the locus of ultimate intersections" is denoted by *l. u. i.*

156. Theorem. In general, *the locus of the ultimate intersections touches each member of the family.* Let I., II., III. be any three members of the family, and let I. and II. intersect at P, and II. and III. at Q. When the curve I. approaches coincidence with II., the point P approaches a definite position on *l. u. i.* of the curves of the family. When the curve III. approaches coincidence with II., Q approaches a definite position on *l. u. i.* When I. and III. both approach coincidence with II., P and Q approach each other along II., and at the same time approach *l. u. i.* When P

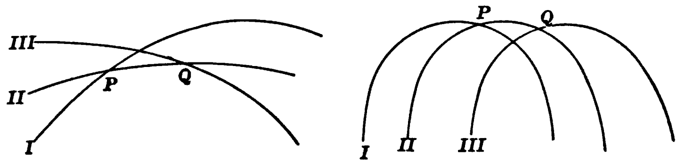

FIG. 94.

and Q finally reach each other on II., they are also on *l. u. i.* Moreover, when P and Q come together, the tangent to II. at P and the tangent to II. at Q come into coincidence as a line which is at the same time a tangent to curve II. and a tangent to *l. u. i.* at the point where P and Q meet. Thus the curve II. and *l. u. i.* have a common tangent at their common point. Similarly it can be shown that *l. u. i.* touches every other curve of the family. Since, in general, each point of *l. u. i.* may be approached in the manner indicated in this article, the above **theorem** may be thus **supplemented:** In general, *l. u. i. is touched at each of its points by some member of the family.*

NOTE 1. The family of circles, Fig. 92, will serve to illustrate this theorem.

NOTE 2. An *analytical proof* of the theorem is given in Art. 157, Note 3.

NOTE 3. It is necessary to use the qualifying phrase *in general* in the enunciation of the theorem, for there are some families of curves (viz. curves having double points and cusps, see Arts. 163, 164), in which a part of *l. u. i.* may not touch any member of the family. It is beyond the scope of this book to go into these cases in detail. (See Edwards, *Treatise on the Diff. Cal.*, Art. 365; Murray, *Differential Equations*, Chap. IV.) Illustrations may be obtained by sketching some curves of the families $(y+c)^2 = x^3$ and $(y+c)^2 = x(x-3)^2$.

157. To find the envelope of a family of curves having one parameter. It is in accordance with the definitions and theorem in Arts. 154–156 to say that *the envelope of a family of curves* $f(x, y, a) = 0$, *if there be an envelope, is*, in general, *the locus of the limiting position of the intersection of any one of the curves of the family,* say the curve

$$f(x, y, a) = 0 \tag{1}$$

with another curve of the family, viz.

$$f(x, y, a + \Delta a) = 0 \tag{2}$$

when the second curve approaches coincidence with the first; that is, when Δa approaches zero.

From (1) and (2), $f(x, y, a + \Delta a) - f(x, y, a) = 0$;

hence
$$\frac{f(x, y, a + \Delta a) - f(x, y, a)}{\Delta a} = 0. \tag{3}$$

Now Equations (1) and (3) may be used, instead of (1) and (2), to find the points of intersection of curves (1) and (2). If $\Delta a \doteq 0$, the point of intersection approaches an ultimate point of intersection. When (Arts. 22, 79) $\Delta a \doteq 0$, Equation (3) becomes

$$\frac{\partial}{\partial a} f(x, y, a) = 0. \tag{4}$$

Thus the coördinates x and y of the point of ultimate intersection of curves (1) and (2) satisfy Equations (1) and (4); and, accordingly, satisfy the relation which is deduced from (1) and (4) by the elimination of a. Hence, *in order to find the equation of l. u. i. of the family of curves* $f(x, y, a) = 0$ *eliminate* a *between the equations*

$$f(x, y, a) = 0 \ \textit{ and } \ \frac{\partial}{\partial a} f(x, y, a) = 0. \tag{5}$$

The result obtained is, *in general,* also the equation of the envelope.

NOTE 1. A slightly different way of making the above deduction is as follows. Let the equations of two curves of the family be

$$f(x, y, a) = 0 \quad (6), \qquad \text{and} \qquad f(x, y, a + h) = 0. \tag{7}$$

By Art. 64, Eq. (3), Equation (7) may be written

$$f(x, y, a) + h\frac{\partial}{\partial a}f(x, y, a + \theta h) = 0, \text{ in which } |\theta| < 1. \tag{8}$$

By virtue of (6) this becomes $\dfrac{\partial}{\partial a}f(x, y, a + \theta h) = 0.$ (9)

Accordingly, the coördinates of the intersection of curves (6) and (7) satisfy (6) and (9). When h becomes zero, the point of intersection becomes an ultimate point of intersection. Hence the ultimate points of intersection satisfy equations $f(x, y, a) = 0$ and $\dfrac{\partial}{\partial a}f(x, y, a) = 0$, and, accordingly, the a-eliminant of these equations.*

NOTE 2. For an interesting and useful derivation of result (5) for cases in which $f(x, y, a)$ is a rational integral function of a, see Lamb's *Calculus*, Art. 157.

NOTE 3. To show that, in general, *the a-eliminant of Equations* (5) *touches any curve of the family.*

Let the second of Equations (5) on being solved for a give $a = \phi(x, y)$. Then the equation of the *l. u. i.* of the family of curves $f(x, y, a) = 0$ is

$$f(x, y, a) = 0 \text{ in which } a = \phi(x, y). \tag{10}$$

The slope $\dfrac{dy}{dx}$ of any one of the family of curves $f(x, y, a) = 0$ is given (see Art. 56), by the equation

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{11}$$

The slope $\dfrac{dy}{dx}$ of the *l. u. i.* is obtained from Equations (10). On taking the total x-derivative in the first of these equations,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial a}\frac{da}{dx} = 0. \tag{12}$$

But by the second of (5), $\dfrac{\partial f}{\partial a} = 0$, and accordingly, (12) reduces to

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{13}$$

Thus the slope of the *l. u. i.* and the slope of any member of the family are both given by the same equation. Hence, at a point common to any curve and the *l. u. i.*, the slopes of both are the same, and accordingly, the curve and the *l. u. i.* touch at that point.

Sometimes the value of $\dfrac{dy}{dx}$ obtained from (11) is indeterminate in form, and the slopes of the curve and *l. u. i.* may not be the same. See Arts. 165, 156 (Note 3), and Lamb, *Calculus*, Art. 158.

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* This method of finding envelopes appears to be due to Leibnitz.

EXAMPLES.

1. Find the envelope of the family of circles (see Art. 154)

$$(x-c)^2 + y^2 = 4. \tag{1}$$

Here, on differentiation with respect to the parameter c,

$$2(x-c) = 0. \tag{2}$$

The elimination of c between these equations gives

$$y^2 = 4,$$

which represents the two straight lines $y = 2$, $y = -2$.

2. Find the envelope of the family of lines

$$y = mx - 2pm - pm^3, \tag{1}$$

in which m is the parameter. (This is the equation of the general normal of the parabola $y^2 = 4px$; see works on analytic geometry.) On differentiation with respect to the parameter m,

$$0 = x - 2p - 3pm^2. \tag{2}$$

The m-eliminant of (1) and (2) is the equation of the envelope.

On taking the value of m in (2) and substituting it in (1), and simplifying and removing the radicals, there is obtained

$$27py^2 = 4(x-2p)^3. \tag{3}$$

Note 4. In Art. 152 it is shown that the normals to a curve touch its evolute. It also appears from Art. 152 that each tangent to an evolute is normal to the original curve. Accordingly, it may be said that *the evolute of a curve is the envelope of its normals*, and likewise that *the evolute of a curve is the l. u. i. of its (family of) normals.* (See Art. 152, Note 2, and Art. 149, Note 5.)

Note 5. Compare Ex. 1, Art. 151, Ex. 2 above, and Ex. 1, Art. 158.

3. If A, B, C are functions of the coördinates of a point and m a variable parameter, show that **the envelope of $Am^2 + Bm + C = 0$ is $B^2 - 4AC = 0$.**

Note 6. *The result in Ex. 3 is the same in form as the condition that the roots of the quadratic equation in m be equal.* This result is immediately applicable in many instances. It is very easily deduced on taking the point of view explained in the article mentioned in Note 2.

4. Deduce the result in Ex. 3 without reference to the calculus. Apply this result to Ex. 1.

N.B. *Make figures for the following examples.*

5. Find the curves whose tangents have the following general equations, in which m is the variable parameter:

(1) $y = mx + a\sqrt{1+m^2}$. (2) $y = mx + \sqrt{a^2m^2 + b^2}$.
(3) $y = mx \pm \sqrt{am^2 + bm + c}$. (4) $y = mx + a\sqrt{m}$.
(5) $m^3x = my + a$. (6) $y - b = m(x-a) + r\sqrt{1+m^2}$.

6. Find the envelopes of the following lines:

(1) $x \sin\theta - y\cos\theta + a = 0$, parameter θ. (2) $x + y\sin\theta = a\cos\theta$, parameter θ. (3) $ax \sec\alpha - by \operatorname{cosec}\alpha = a^2 - b^2$, parameter α.

7. Find the envelopes of (1) the parabolas $y^2 = 4a(x-a)$, parameter a; (2) the parabolas $cy^2 = a^2(x-a)$, parameter a.

8. Show that if A, B, C are functions of the coördinates of a point, and α a variable parameter, the envelope of $A\cos\alpha + B\sin\alpha = C$ is $A^2 + B^2 = C^2$.

9. Find the evolute of the ellipse $x = a\cos\phi$, $y = b\sin\phi$, considering the evolute of a curve as the envelope of its normals.

10. One of the lines about a right angle passes through a fixed point, and the vertex of the angle moves along a fixed straight line: find the envelope of the other line.

11. From a fixed point on the circumference of a circle, chords are drawn, and on these as diameters circles are described. Show that they envelop a cardioid.

158. To find the envelope of a family of curves having two parameters. Let

$$f(x, y, a, b) = 0$$

be a family of curves which has two parameters. If there is a given relation between these parameters, say

$$F(a, b) = 0,$$

then the two parameters practically come to one, and accordingly, the case reduces to that considered in Art. 157.

EXAMPLES.

1. Find the envelope of the normals to the parabola $y^2 = 4px$. The equation of the normal at any point (x_1, y_1) on this parabola is

$$y - y_1 + \frac{dx_1}{dy_1}(x - x_1) = 0.$$

This reduces to $$2py - 2py_1 + xy_1 - x_1y_1 = 0. \qquad (1)$$

Here there are two parameters, x_1 and y_1. They are connected by the relation

$$y_1^2 = 4px_1.$$

Hence (1) becomes $$2py - 2py_1 + xy_1 - \frac{y_1^3}{4p} = 0, \qquad (2)$$

which involves only a single parameter y_1. On differentiating in (2) with respect to the parameter y_1 and then eliminating y_1, there will appear the equation of the envelope, viz.

$$27py^2 = 4(x - 2p)^3.$$

Compare Ex. 1 with Ex. 1, Art. 151, and Ex. 2, Art. 157.

NOTE. This problem may be expressed: Find the envelope of the line (1), given that the point (x_1, y_1) moves along the parabola $y^2 = 4px$.

2. Find the envelope of the line

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (1)$$

when the sum of its intercepts on the axes is always equal to a constant c.

Since $$a + b = c, \qquad (2)$$

Equation (1) may be written $\dfrac{x}{a} + \dfrac{y}{c-a} = 1$,

i.e. $$(c-a)x + ay = ac - a^2. \qquad (3)$$

Thus (1) is transformed into an equation involving a single parameter a. On differentiating in (3) with respect to the parameter a,

$$-x + y = c - 2a. \qquad (4)$$

The elimination of a between (3) and (4) gives

$$x^2 + y^2 + c^2 = 2cx + 2xy + 2cy.$$

This reduces to $$\sqrt{x} + \sqrt{y} = \sqrt{c}.$$

See Ex. 7, Art. 59.

The elimination of a and b can also be performed thus:

Differentiation in (1) and (2) with respect to a gives

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \text{ and } 1 + \frac{db}{da} = 0.$$

On equating the values of $\dfrac{db}{da}$,

$$\frac{x}{a^2} = \frac{y}{b^2}; \text{ whence } \frac{b}{a} = \frac{\sqrt{y}}{\sqrt{x}}. \qquad (5)$$

From (2) and (5), $a = \dfrac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$, $b = \dfrac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

On substitution in (1) and reduction, $\sqrt{x}+\sqrt{y}=\sqrt{c}$.

This second method is generally more useful than that used in Ex. 1 and in the first way of working Ex. 2, in cases when the two parameters are involved symmetrically in the equation and in the expression of the relation between the parameters.

3. Find the envelope of the straight lines the product of whose intercepts on the axes of coördinates is equal to a^2.

4. Find the envelope of a straight line of fixed length a which moves with its extremities in two lines at right angles to each other.

5. A set of ellipses which have a common centre and axes, and in which the sum of the semi-axes is equal to a constant a, is drawn: find the envelope of the ellipses.

6. Show that the envelope of a family of co-axial ellipses having the same area consists of two conjugate rectangular hyperbolas.

7. Circles are described on the double ordinates of the parabola $y^2 = 4ax$ as diameters: show that the envelope is the equal parabola $y^2 = 4a(x+a)$.

8. Circles are described having for diameters the double ordinates of the ellipse whose semi-axes are a and b: show that their envelope is the co-axial ellipse whose semi-axes are $\sqrt{a^2+b^2}$ and b.

9. About the points on a fixed ellipse as centre, ellipses are described having axes equal and parallel to the axes of the fixed ellipse: show that their envelope is an ellipse whose axes are double those of the fixed ellipse.

10. A straight line moves so that the sum of the squares of the perpendiculars on it from two fixed points $(\pm c, 0)$ is constant $(= 2k^2)$: show that its envelope is the conic $\dfrac{x^2}{k^2-c^2}+\dfrac{y^2}{k^2}=1$.

11. If the difference of the squares in Ex. 10 is constant, show that the envelope is a parabola.

12. Show that if the corner of a rectangular piece of paper be folded down so that the sum of the edges left unfolded is constant, the crease will envelop a parabola.

ASYMPTOTES.

159. Rectilinear asymptotes. In preceding studies acquaintance has been made with two lines related to the hyperbola, called asymptotes and possessing the following properties: (a) These lines are the limiting positions which the tangents to the hyperbola approach when the points of contact recede for an

infinite distance along the curve (or, as it may be expressed, recede towards infinity); (*b*) the lines themselves do not lie altogether at infinity. (This is the mathematical way of saying that the lines run across the field of view; in fact, in the case of the hyperbola they pass through the centre of the curve.)

Besides hyperbolas there are many other curves which have branches extending to an infinite distance and which have associated with them certain lines having properties like (*a*) and (*b*); namely, lines: (1) that are the limiting positions which the tangents to the infinite branches approach when the points of contact recede towards infinity; (2) that do not lie altogether at infinity; for instance, using rectangular coördinates, lines that pass within a finite distance of the origin.

Lines having properties (1) and (2) are called *asymptotes* of the curves. Thus an ellipse cannot have an asymptote, since it has no branch extending to infinity (see Ex. 3, Art. 161). Again the parabola $y^2 = 4px$ has no asymptote, for (see Ex. 4, Art. 161) the tangent at an infinitely distant point of this parabola crosses each of the axes of coördinates at an infinite distance from the origin, and, accordingly, no part of this tangent can be in sight; *i.e.* it lies wholly at infinity. (The asymptotes are apparent in the figures on pages 410–414.)

It will now be shown how an examination may be made for the asymptotes of curves whose equations have the form

$$F(x, y) = 0, \tag{1}$$

where $F(x, y)$ is a rational integral function of x and y. For this it is necessary to call to mind the algebraic property stated in the following note,

Algebraic Note. On substituting $\frac{1}{t}$ for x in the rational integral equation

$$c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n = 0, \tag{a}$$

and clearing of fractions, it becomes

$$c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1} + c_nt^n = 0. \tag{b}$$

It is shown in algebra that if a root of Equation (*b*) approaches zero, c_0 approaches zero; and that if a second root also approaches zero, c_1 also approaches zero. But, since $x = \frac{1}{t}$, when a root of (*b*) approaches zero, a

root of (a) increases beyond all bounds, *i.e.*, to use a common phrase, it approaches infinity. Hence, the condition that a root of (a) approach infinity is that c_0 approach zero, and the condition that a second root of (a) at the same time approach infinity is that c_1 also approach zero; and so on for other roots approaching infinity. This is briefly expressed by saying that equation (a) has a root equal to infinity when $c_0 = 0$, and has two roots equal to infinity when $c_0 = 0$ and $c_1 = 0$.

160. To find asymptotes which are parallel to the axes of coördinates. Suppose that the equation of the curve $F(x, y) = 0$ [Art. 159 (1)] is of the nth degree, and that the terms in the first member of this equation are arranged according to decreasing powers of y. Then the equation has the form

$$p_0y^n + p_1y^{n-1} + p_2y^{n-2} + \cdots + p_{n-1}y + p_n = 0. \tag{1}$$

Here, p_0 is a constant; p_1 may be an expression in x of the first degree at most, say $ax + b$; p_2 may be of the second degree at most, say $cx^2 + dx + e$; p_3 may be of the third degree in x at most; $\cdots$; and p_n may be of the nth degree in x at most. For if any one of the respective p's were of a higher degree than that specified above, $F(x, y)$ would be of a higher degree than the nth.

Ex. 1. Arrange the first members of the following equations (a) in descending powers of x; (b) in descending powers of y:

(1) $xy - ay - bx = 0.$ (2) $x^3 + xy^2 + 2x^2 - 2y^2 - 7x + 4y - 11 = 0.$

(3) $2xy^2 - x^2y + 3y^2 - 3x^2 + 4xy - 2x + 7y + 1 = 0.$

(4) $y^3 + x^2y + x^2 + 2xy + 7x + 2 = 0.$

Now suppose that in (1) $p_0 = 0$; then (1) may be written

$$0 \cdot y^n + (ax + b)y^{n-1} + (cx^2 + dx + e)y^{n-2} + p_3y^{n-3} + \cdots$$
$$+ p_{n-1}y + p_n = 0. \tag{2}$$

If this be regarded as an equation of the nth degree in y, then to *any* finite value of x there correspond n values of y, one of which is infinitely great. If also $ax + b = 0$, *i.e.* if $x = -\frac{b}{a}$, a second of the n values of y is infinitely great. In a similar way points whose abscissas are infinitely great and whose ordinates are finite may be found.

Ex. 2. Thus in Ex. 1 (1) the equation, which is of the *second* degree, may be written $y(x-a)-bx=0$. Accordingly one value of y is infinite; a second value of y is infinite when $x=a$.

Ex. 3. Show that a second value of x is infinite when $y=b$.

It will now be shown that *an infinite ordinate whose distance from the origin is finite is tangent to the curve at the infinitely distant point.*

On differentiating in (2) with respect to x and solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx}=-\frac{ay^{n-1}+(2\,cx+d)y^{n-2}+\cdots+p'_n}{(n-1)(ax+b)y^{n-2}+(n-2)(cx^2+dx+e)y^{n-3}+\cdots+p_{n-1}}. \quad (3)$$

When $x=-\frac{b}{a}$, the numerator in the second member is an infinity of an order at least two higher than the denominator, and hence the value of the fraction is then infinite. Hence the line $x=-\frac{b}{a}$ is a tangent at any point for which $x=-\frac{b}{a}$ and $y=\infty$.

In a similar way it can be shown that if one of the values of x in Equation (1), Art. 159, is infinite when $y=c$, in which c is finite, then $y=c$ is a tangent at any point for which $x=\infty$ and $y=c$.

NOTE 1. If [see Eq. (2)] $x=-\frac{b}{a}$ also satisfies $cx^2+dx+e=0$, then *three* values of y in $F(x, y)=0$ are infinitely great for this value of x. The line $x=-\frac{b}{a}$ is then an inflexional tangent (see Art. 78, Note 1) at infinity.

NOTE 2. This method of finding asymptotes parallel to the axes can be applied to curves whose equations are not of the kind considered above. Instances are given in Exs. 7, 8 (6), (9) that follow.

EXAMPLES.

4. Find the asymptotes of the curves in Ex. 1.

5. Determine the finite points (if they exist) in which each asymptote in Ex. 4 meets the curve to which it belongs.

6. Show that the line $x=a$ is an asymptote of the curve $y=\frac{\phi(x)}{x-a}$ when $\phi(a)$ and $\phi'(a)$ are finite.

Here, $\lim_{x\doteq a} y=\infty$. Also $\frac{dy}{dx}=\frac{(x-a)\phi'(x)-\phi(x)}{(x-a)^2}$; whence $\lim_{x\doteq a}\frac{dy}{dx}=\infty$. Hence $x=a$ is a tangent at an infinitely distant point ($x=a$, $y=\infty$).

7. Examine $y=\tan x$ for asymptotes.

Here $y=+\infty$ when $x=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$.

Also, $\frac{dy}{dx}=\sec^2 x$. Hence $\frac{dy}{dx}=\infty$ when $x=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$.

$\therefore x=\frac{\pi}{2}$, $x=\frac{3\pi}{2}$, $x=\frac{5\pi}{2}$, $\cdots$, are asymptotes.

8. Determine the asymptotes of the following curves: (1) The hyperbola $xy = a^2$. (2) The cissoid $y^2 = \frac{x^3}{2a - x}$. (3) The witch $y = \frac{8a^3}{x^2 + 4a^2}$. (4) $(x^2 - a^2)(y^2 - b^2) = a^2b^2$. (5) $a^2x = y(x-a)^2$. (6) $y = \log x$. (7) $y = e^x$. (8) The probability curve $y = e^{-x^2}$. (9) $y = \sec x$.

161. Oblique asymptotes. There are asymptotes which are not parallel to either axis. The method of finding them can best be shown by an example.

EXAMPLES.

1. Find the asymptotes of the folium of Descartes (see page 413)

$$x^3 + y^3 = 3axy. \tag{1}$$

First find the intersections of this curve and the line

$$y = mx + b. \tag{2}$$

On solving these equations simultaneously,

$$(1 + m^3)x^3 + 3(m^2b - am)x^2 + 3(mb^2 - ab)x + b^3 = 0.$$

Line (2) is a tangent to the curve (1) at an infinitely distant point, if two roots of this equation are infinitely great. That is, if

$$1 + m^3 = 0, \text{ and } m^2b - am = 0. \tag{3}$$

That is, on solving Equations (3) for m and b, if

$$m = -1, \text{ and } b = -a.$$

Hence, the asymptote is $\qquad y + x + a = 0.$

NOTE 1. *A curve whose equation is of the nth degree has n asymptotes, real or imaginary.* This may be apparent from the preceding discussion. For proof of this theorem see references for collateral reading, Art. 162.

In Ex. 1 two values of m in Equations (3) are imaginary; thus curve (1) has one real and two imaginary asymptotes.

2. Find the asymptotes of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

3. Show by the method used in Ex. 1 that the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ has no real asymptotes.

4. Show by the method used in Ex. 1 that the parabola $y^2 = 4px$ does not have an asymptote.

5. Find the asymptotes of the following curves: (1) $y^3 = x^3 + x$. (2) $x^4 - y^4 - 3x^3 - xy^2 - 2x + 1 = 0$. (3) $xy(y - x) = 3x^2 + 2y^2$. (4) $(x^2 - y^2)^2 - 4y^2 + y + 2x + 3 = 0$. (5) $x^3 - 8y^3 + 3x^2 - xy - 2y^2 = 0$.

Note 2. **Other methods of finding asymptotes.**

a. Find the values of the intercepts on the axes of coördinates of the tangent at a point (x', y') on a curve [see Art. 59, Note 3 (1)], when $x' = \infty$, or $y' = \infty$, or both x' and y' are infinitely great. If one or both of these intercepts is finite, the tangent is an asymptote. Its equation can be written on finding its intercepts.

6. Apply this method to Exs. 2, 4, above.

b. Find the length of the perpendicular from the origin to the tangent at (x', y') when $x' = \infty$, or $y' = \infty$, or both x' and y' are infinitely great. If this length is finite, the tangent is an asymptote.

7. Do Exs. 2, 4, by this method.

c. By means of the equation of the curve express y in terms of a series in decreasing powers of x, or express x in terms of a series in decreasing powers of y. From one of these expressions there may sometimes be deduced the equation of a straight line which, for infinitely distant points, closely approximates to the equation of the curve.

8. Thus, in the hyperbola in Ex. 2,

$$y = b\sqrt{1 + \frac{x^2}{a^2}} = \pm \frac{bx}{a}\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}}$$

$$= \pm \frac{bx}{a}\left(1 - \frac{1}{2}\frac{a^2}{x^2} - \cdots\right) = \pm \frac{bx}{a} \mp \frac{ba}{x} \mp \frac{1}{4}\frac{ba^3}{x^3} \pm \cdots.$$

It is apparent from this that the farther away the points on the lines $y = \pm \frac{bx}{a}$ are taken, the more nearly will they satisfy the equation of the hyperbola, and that when x increases beyond all bounds, the points on these lines satisfy the equation of the hyperbola. Accordingly, these lines are asymptotes.

Note 3. **Curvilinear asymptotes.** Expansion may sometimes reveal the equation of a curve of higher degree than the first whose infinitely distant points also satisfy the equation of the given curve. Accordingly the two curves coincide at infinitely distant points. The two curves are said to be asymptotic, and the new curve is called *a curvilinear asymptote* of the original curve. For a discussion on curvilinear asymptotes see Frost's *Curve Tracing*, Chaps. VII. and VIII.

162. Rectilinear asymptotes: polar coördinates. In order to find the asymptotes of the curve

$$f(r, \theta) = 0 \tag{1}$$

a method similar to that outlined in Art. 161, Note 2 (*b*), can be used. First find the value of θ in Equation (1) for which the radius vector r is infinitely great. Suppose that this value of θ is θ_1. Thus the point ($r=\infty$, $\theta=\theta_1$) is an infinitely distant point of the curve. If the tangent TN at this infinitely distant point is an asymptote, it passes within a finite distance from O. Accordingly, TN is parallel to the radius vector, and the subtangent OM, viz. $r^2\dfrac{d\theta}{dr}$ (Art. 61) is finite for ($r=\infty$, $\theta=\theta_1$).

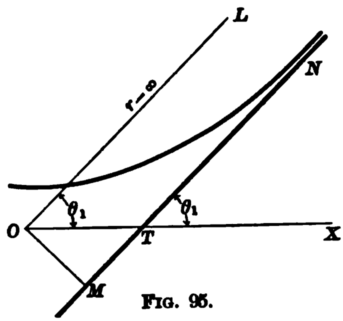

FIG. 95.

EXAMPLES.

1. Find and draw the asymptote to the reciprocal spiral $r\theta = a$.

Here $\qquad r = \dfrac{a}{\theta}. \quad \therefore r \doteq \infty$ when $\theta \doteq 0$.

Also $\qquad \theta = \dfrac{a}{r}. \quad \therefore \dfrac{d\theta}{dr} = -\dfrac{a}{r^2}. \quad \therefore r^2\dfrac{d\theta}{dr} = -a.$ (See Fig., page 414.)

Hence the asymptote is parallel to the initial line and at a distance a to the *left* of one who is looking along the initial line in the positive direction.

NOTE 1. The *convention* used in Ex. 1 is as follows: A *positive subtangent* is measured to the *right* of a person who may be looking along the infinite radius vector in its positive direction, and a *negative subtangent* is measured toward the *left*.

2. Find and draw the asymptotes to the following curves: (1) $r\sin\theta = a\theta$. (2) $r\cos\theta = a\cos 2\theta$. (3) $r\sin\dfrac{\theta}{2} = a$.

NOTE 2. **Circular asymptotes.** If the radius vector r approaches a fixed limit, a say, when θ increases beyond all bounds, then as θ increases, the curve approaches nearer to coincidence with the circle whose centre is at the pole and whose radius is a. This circle, whose equation is $r = a$, is said to be *a circular asymptote*, or *the asymptotic circle*, of the curve.

3. In the reciprocal spiral, Ex. 1, if $\theta \doteq \infty$, then $r \doteq 0$. Hence the asymptotic circle is a circle of zero radius, viz. the pole.

4. Find the rectilinear and the circular asymptote of $r = \dfrac{\theta}{\theta - 1}$.

References for collateral reading on asymptotes. McMahon and Snyder's *Diff. Cal.*, Chap. XIV., pages 221–242; F. G. Taylor's *Calculus*, Chap. XVI., pages 228–249, and Edwards's *Treatise on the Differential Calculus*, Chap. VIII., pages 182–210, contain interesting discussions on asymptotes, with many illustrative examples. For a more extended account of asymptotes see Frost's *Curve Tracing*, Chaps. VI.–VIII., pages 76–129.

Singular Points.

163. Singular points. On some curves there are particular points at which the curves have certain peculiar properties which they do not possess at their points in general. For instance, there are points of maximum or minimum ordinates (Art. 75), points of inflexion (Art. 78), and points of undulation (Art. 78). There are also points through which a curve passes twice or more than twice (see Figs. 96 *a*, *b*, *c*), and at which it has two or more different tangents; there are points through which pass two branches of a curve that have a common tangent (Figs. 97 *a*, *b*, *c*, *d*); and there are other peculiar points hereafter described. Points of maximum and minimum ordinates depend on the relative position of a curve and the axes of coördinates; the peculiarities at the other points referred to above are independent of the axes and belong to the curve whatever be its situation. Points at which a curve has peculiarities of this kind are called *singular points*. Some of these singular points are considered in Arts. 164, 165.

164. Multiple points. Double points. Cusps. Isolated points. Multiple points are those through which a point moving along the curve, while changing the direction of its motion continuously, can pass two or more times, and at which the curve may have two or more different tangents.

For example, in moving from L to M along the curves in Figs. 96 *a*, *b*, *c*, a point passes through A and C three times and through B and D twice. At A there are three different tangents, at C there are three, and at B and D there are two each. Points, such

as B and D, through which the point moving along the curve, while continuously changing the direction of its motion, can pass

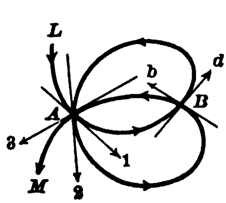

Fig. 96 *a.*

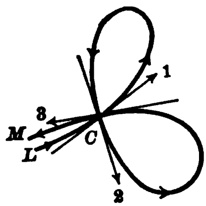

Fig. 96 *b.*

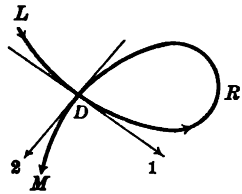

Fig. 96 *c.*

twice, are called *double points;* points such as A and C are called *triple points.* The curve $r = a \sin 2\theta$ (see p. 414) has a quadruple point.

Note 1. Multiple points are also called *nodes.* (Latin *nodus*, a knot.)

Cusps are points where two branches of the curve have the same tangent. See Figs. 97 *a, b, c, d.*

In Fig. 97 a both branches of the curve stop at A and lie on opposite sides of their common tangent at A. In Fig. 97 b both branches stop at B and lie on the same side of the tangent at B. Both branches of the curve pass through C. Accordingly C is sometimes called a double cusp. If a point is moving along a curve LKM which has a single cusp at K (Fig. 97 d), there is an

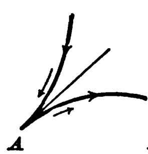

Fig. 97 *a.*

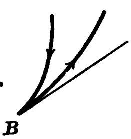

Fig. 97 *b.*

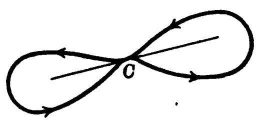

Fig. 97 *c.*

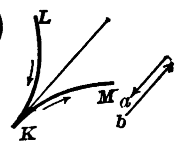

Fig. 97 *d.*

abrupt (or discontinuous) change made in the direction of its motion on its passing through K. On arriving at K from L the moving point is going in the direction a; on leaving K for M the moving point is going in the direction b. Thus at K it has suddenly changed the direction of its motion by the angle π.

Note 2. A cusp such as K (Fig. 97 d) may be supposed to be the final (or limiting) condition of a double point like D (Fig. 96 c) when the loop DR dwindles to zero and the two tangents at D become coincident.

Isolated or **conjugate points** are individual points which satisfy the equation of the curve but which are isolated from (*i.e.* at a finite distance from) all other points satisfying the equation.

EXAMPLES.

1. Sketch the curve $y^2 = (x-a)(x-b)(x-c)$, in which a, b, and c, are positive and $a < b < c$.

2. Sketch the curve $y^2 = (x-a)(x-b)^2$, in which $a < b$ and both are positive.

3. Sketch the curve $y^2 = (x-a)^2(x-b)$, in which a and b are as in Ex. 2.

4. Sketch the curve $y^2 = (x-a)^3$, in which a is positive.

The sketch in Ex. 1 will show an oval from $x = a$ to $x = b$, a blank space from $x=b$ to $x=c$, and a curve extending from $x=c$ to the right. The sketch in Ex. 2 will show a curve having a double point at $(b, 0)$. The sketch in Ex. 3 will show a conjugate point at $(a, 0)$, a blank space from $x=a$ to $x=b$, and a curve extending from $x=b$ to the right. The sketch in Ex. 4 will show a curve having a cusp at $(a, 0)$.

NOTE 3. **Other singular points.** There also are points called **salient points,** like D (Fig. 98), for instance, where two branches of the curve stop but do not have a common tangent. In these cases the slope of the tangent changes abruptly. Accordingly, if $y = \phi(x)$ be the equation of the curve, $\phi'(x)$ is discontinuous at the salient points. (See Exs. 5, 6, below.) A salient point such as D may be considered to be the limiting condition of a double point like D (Fig. 96 *c*), when the loop DR dwindles to zero but the two tangents at D do not become coincident. (Compare Note 2.)

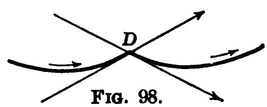

FIG. 98.

FIG. 99.

There are also **stop points,** as A, Fig. 99, where the curve stops and has but one branch. See Ex. 7.

5. In the curve $y(1 + e^{\frac{1}{x}}) = x$ show that when x approaches the origin from the positive side, the slope is zero; if from the negative side, the slope is 1. The origin is thus a salient point. $\left[\text{Suggestion: The slope at the origin may be taken as } \lim_{x \doteq 0} \frac{y}{x}.\right]$ Find the angle between the branches at the origin.

6. In the curve $y = x\dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$ show that when x approaches the origin from the positive side the slope is $+1$, and if from the negative side, the slope is -1. The origin thus is a salient point: find the angle between the branches there.

7. Show that the origin is a stop point in the curve $y = x \log x$.

165. To find multiple points, cusps, and isolated points. From Art. 164 it is evident that in order to determine the character of a point on a curve, it is first of all necessary to examine the tangent (or tangents) there. Let the equation of the curve be

$$f(x, y) = 0, \tag{1}$$

and let $f(x, y)$ be a rational integral function of x and y. Then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \quad \text{[Art. 84, (4).]} \tag{2}$$

Now at a multiple point or a cusp $\frac{dy}{dx}$ has not a *single* definite value, and, accordingly, at such points $\frac{dy}{dx}$ in (2) must have an indefinite form, viz. the form $\frac{0}{0}$.* Hence, at a multiple point of curve (1)

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0. \tag{3}$$

The solutions of Equations (3) will indicate the points which it is necessary to examine.† At these points

$$\frac{dy}{dx} = \frac{0}{0}; \tag{4}$$

the indefinite form in the second member can be evaluated by the method explained in the Appendix, Note C, and applied in Note below.‡ Suppose that the second member of (4) has been evaluated and the resulting equation solved for $\frac{dy}{dx}$. Then: If $\frac{dy}{dx}$ has two real and different values at the point under consideration, the point is a double point or a salient point; if $\frac{dy}{dx}$ has three real and different values there, it is a triple point; and so on. If $\frac{dy}{dx}$

---

* This is frequently called an "*indeterminate*" form. The evaluation of (so-called) "*indeterminate forms*" is discussed in the Appendix, Note C.

† The values of x and y that satisfy Equations (3), may give points that are not on the curve. Of course these points need not be examined further.

‡ Or by other methods referred to in Appendix, Note C.

has two real and equal values at the point which is being examined, the point is a cusp. If $\frac{dy}{dx}$ has imaginary values at the point, it is an isolated point.

If the point is a cusp, *the kind of cusp* can be found by further examination of the curve in the neighborhood of the point. For example, if (x_1, y_1) is known to be a cusp and it is found that for $x = x_1 - h$ (h being infinitesimal), y is imaginary, then the curve does not extend through (x_1, y_1) to the left, and thus the cusp is not a double cusp. If for $x = x_1 + h$, the value of the ordinate of the tangent at (x_1, y_1) is less than the ordinates of both branches of the curve, the cusp is as in Fig. 100. In a similar way tests may be devised and applied in special cases as they arise.

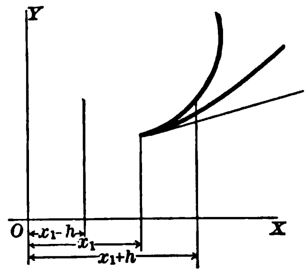

FIG. 100.

NOTE. The evaluation of the second member of Equation (2) gives, by Appendix, Note C, and Art. 81, (5)

$$\frac{dy}{dx} = -\frac{\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y\,\partial x}\dfrac{dy}{dx}}{\dfrac{\partial^2 f}{\partial x\,\partial y} + \dfrac{\partial^2 f}{\partial y^2}\dfrac{dy}{dx}}. \tag{5}$$

If the second member of (5) is not indefinite in form, this equation, on clearing of fractions and combining, becomes

$$\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + 2\frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dx} + \frac{\partial^2 f}{\partial x^2} = 0, \tag{6}$$

a quadratic equation in $\frac{dy}{dx}$. By the theory of quadratic equations, the two values of $\frac{dy}{dx}$ are real and different, real and equal, or imaginary, according as $\left(\frac{\partial^2 f}{\partial y \partial x}\right)^2$ is respectively greater than, equal to, or less than $\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 f}{\partial x^2}$. Hence, the point is a double point, a cusp, or a conjugate point, according as

$$\left(\frac{\partial^2 f}{\partial y \partial x}\right)^2 >, =, \text{ or } < \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 f}{\partial x^2}.$$

If the second member of (5) also is indefinite in form, proceed as required by Note C, remembering that $\frac{dy}{dx}$ here is constant. The resulting equation will be of the third degree in $\frac{dy}{dx}$.

EXAMPLES.

1. Examine the curve $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$ for singular points.

Here
$$\frac{dy}{dx} = -\frac{3x^2 - 14x + 15}{-2y + 4}. \tag{1}$$

On giving each member the indefinite form $\frac{0}{0}$, and solving the equations

$$3x^2 - 14x + 15 = 0,$$
$$-2y + 4 = 0,$$

it results that $x = 3$ or $\frac{5}{3}$, and $y = 2$.

Substitution in the equation of the curve shows that $x = \frac{5}{3}$, $y = 2$, do not satisfy the equation, and that $x = 3$, $y = 2$ do. Accordingly, the point (3, 2) is the point to be further examined.

On evaluating, by the method shown in the Appendix, the second member of (1) for the values $x = 3$, $y = 2$, it is found that

$$\frac{dy}{dx} = -\frac{6x - 14}{-2\frac{dy}{dx}}; \text{ whence } \left(\frac{dy}{dx}\right)^2 = 2, \text{ and } \frac{dy}{dx} = \pm\sqrt{2}.$$

Thus the curve has a double point at (3, 2), and the slopes of the tangent there are $+\sqrt{2}$ and $-\sqrt{2}$.

[The curve consists of an oval between the points (1, 2), and (3, 2), and two branches extending to infinity to the right of (3, 2).]

2. Sketch the curve in Ex. 1.

3. Examine the following curves for singular points:

(1) $a^2y^2 = x^2(a^2 - x^2)$. (2) $x^3 + 9x^2 - y^2 + 27x + 2y + 26 = 0$.

(3) $y^3 - x^2 - 3y^2 + 3y + 4x - 5 = 0$. (4) The curve in Ex. 5 (5), Art. 161.

(5) $x^3 + y^3 + 3x^2y + 3xy^2 - 10y^2 - 16xy - 10x^2 + 25x + 29y - 28 = 0$.

(6) $x^3 - y^2 - 10x^2 + 33x - 36 = 0$.

166. Curve tracing. Some of the matters involved in curve tracing have been discussed in Arts. 75–78, 159–165. To do more than this is beyond the scope of a primary text-book on the calculus. The topic is mentioned here merely for the purpose of giving a few exercises whose solutions require the simultaneous application of methods for finding points of maximum and minimum, asymptotes, and singular points.

NOTE 1. For a fuller elementary treatment of singular points and curve tracing, see McMahon and Snyder, *Diff. Cal.*, Chaps. XVII., XVIII., pp. 275–306; F. G. Taylor, *Calculus*, Chaps. XVII., XVIII., pp. 250–278; Edwards, *Treatise on Diff. Cal.*, Chaps. IX., XII., XIII.; Echols, *Calculus*, Chaps. XV., XXXI., pp. 147–164, 329–346. The classic English work on the subject is Frost's *Curve Tracing* (Macmillan & Co.), a treatise which is highly praised both from the theoretical and the practical point of view.*

NOTE 2. For the application of the calculus to the study of surfaces (their tangent lines and planes, curvature, envelopes, etc.) and curves in space, see Echols, *Calculus*, Chaps. XXXII.–XXXV., pp. 347–390, and the treatises of W. S. Aldis and C. Smith on *Solid Geometry*.

EXAMPLES.

1. Trace the curves in Ex. 8, Art. 160; in Ex. 5, Art. 161; in Ex. 2, Art. 162; in Ex. 3, Art. 165.

2. Trace the following curves:

(1) $y^2 = x^4(1 - x^2)$. (2) $y^2 = x^2(1 - x)$. (3) $x^4 - 4x^2y - 2xy^2 + 4y^2 = 0$. (4) $2y^2 = 4xy - x^3$. (5) $r = a\cos 4\theta$.

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* A recent important work on curves is Loria's *Special Plane Curves*, a German translation of which (xxi. + 744 pp.) is published by B. G. Teubner, Leipzig.

CHAPTER XIX.

INFINITE SERIES.

EXPANSION OF FUNCTIONS IN INFINITE SERIES. INTEGRATION AND DIFFERENTIATION OF INFINITE SERIES. EXPANSIONS OBTAINED BY INTEGRATION AND DIFFERENTIATION.

N.B. There are some students whose time is limited and who require to obtain as speedily as may be a working knowledge of Taylor's and Maclaurin's expansions. These students had better proceed at once to **Arts. 175, 180,** work **the examples in Arts. 176 and 178,** and then take up **Art. 174.** It is, perhaps, advisable in any case to do this before reading this chapter and the other articles in Chapter XX. Those who are studying the calculus as a "culture" subject should become acquainted with the ideas and principles described, or referred to, in Chapters XIX., XX. A thorough understanding of these ideas and principles is *absolutely essential* for any one who intends to enter upon the study of higher mathematics.

167. Infinite series: definitions, notation. An infinite series consists of a set of quantities, infinite in number, which are connected by the signs of addition and subtraction, and which succeed one another according to some law. A few infinite series of a simple kind occur in elementary arithmetic and algebra.

For instance, the geometrical series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots; \tag{1}$$

the geometrical series

$$1 + x + x^2 + \cdots + x^{n-1} + x^n + x^{n+1} + \cdots, \tag{2}$$

which may also be obtained by performing the division indicated in $\frac{1}{1-x}$; the geometrical series

$$1 - x + x^2 + \cdots + (-1)^n x^{n-1} + \cdots, \tag{3}$$

which may also be obtained by performing the division indicated in $\frac{1}{1+x}$; the geometrical series

$$a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + ar^{n+1} + \cdots; \tag{4}$$

the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots. \tag{5}$$

The successive quantities in an infinite series, beginning with the first quantity, are usually denoted by

$$u_0,\ u_1,\ u_2,\ \cdots,\ u_{n-1},\ u_n,\ u_{n+1},\ \cdots;$$

or, in order to show a variable, x say, by

$$u_0(x),\ u_1(x),\ u_2(x),\ \cdots,\ u_{n-1}(x),\ u_n(x),\ u_{n+1}(x),\ \cdots.$$

Then the series is

$$u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + u_{n+1} + \cdots. \tag{6}$$

The *value of the series* is often denoted by s; and the symbol s_n is generally used to denote the sum or value of the series obtained by taking the first n terms of the infinite series; thus,

$$s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}.$$

The *value of the infinite series* (6) is **the limit** *of the sum of the quantities in the series; i.e.* the value of the series is the limit of the sum of n terms of the series when n increases beyond all bounds.* This is expressed in mathematical symbols

$$s = \lim_{n\doteq\infty} s_n. \tag{7}$$

(This limit s is frequently, but not quite correctly, called "the sum of the series" or "the sum of the series to infinity.")

Thus, in (1), $\quad s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 2\left(1 - \frac{1}{2^{n-1}}\right),$

and hence $\quad s = \lim_{n\doteq\infty} s_n = 2;$ (7)

in (2), $\quad s_n = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1},$

and hence $\quad s = \lim_{n\doteq\infty} s_n = \infty$ when $x \geqq 1$ and $x \leqq -1,$ (8)

$\quad = \frac{1}{1-x}$ when $-1 < x < 1.$ (9)

168. Questions concerning infinite series. The subject of *infinite series* is highly important in mathematics. Such questions as the following arise and require to be answered:

(*a*) Under what conditions may infinite series be employed in mathematical investigation and used in practical work?

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* Thus s is *not* the sum of an infinite number of terms of the series, but is the *limiting value* of that sum.

(*b*) Under what conditions may an infinite series be used to define a function or employed to represent a function?

Thus, in Art. 167, result (8) shows that series (2) *does not* represent the function $\frac{1}{1-x}$ when x is greater than 1 or less than -1 or equal to 1 or -1. This is obvious on a glance at the series; in fact, the greater the number of terms of (2) that are taken, the greater is the error committed in taking the series to represent the function. (For instance, put $x = 2$; then the function is -1 and the series is $+\infty$.) On the other hand, the infinite series (2) *does* represent the function $\frac{1}{1-x}$ when x lies between -1 and $+1$; the greater the number of terms that are taken, the more nearly will the sum of these terms come to the value of the function. The *limit* of the sum of these terms when the number of them is infinite is the function.

(*c*) May two infinite series be added like two finite series? In other words, if

$$u = u_0 + u_1 + u_2 + \cdots$$

and

$$v = v_0 + v_1 + v_2 + \cdots,$$

is

$$u + v = u_0 + v_0 + u_1 + v_1 + \cdots \tag{1}$$

a true equation; and under what conditions is (1) a true equation?

(*d*) May two infinite series be multiplied together like two finite series? In other words, u and v being as in (*c*), is

$$uv = u_0v_0 + u_0v_1 + u_1v_0 + u_1v_1 + u_1v_2 + u_2v_1 + \cdots \tag{2}$$

a true equation; and under what conditions is (2) a true equation?

(*e*) May the principles of Art. 31 and Art. 104 *A*, namely, that the derivative and the integral of the sum of a *finite* number of terms are respectively equal to the sum of the derivatives and the sum of the integrals of these terms (to a constant), be extended to *infinite* series? That is, u_0, u_1, u_2, $\cdots$, being functions of x, if

$$s = u_0 + u_1 + u_2 + \cdots,$$

are

$$\int_a^\beta s\,dx = \int_a^\beta u_0\,dx + \int_a^\beta u_1\,dx + \int_a^\beta u_2\,dx + \cdots, \tag{3}$$

and

$$\frac{d}{dx}(s) = \frac{d}{dx}(u_0) + \frac{d}{dx}(u_1) + \frac{d}{dx}(u_2) + \cdots, \tag{4}$$

true equations; and what are the conditions which must be satisfied in order that these equations be true? Equations (3) and (4) may be expressed:

$$\int_a^\beta \left[\lim_{n\doteq\infty} s_n(x)\right]dx = \lim_{n\doteq\infty}\left[\int_a^\beta s_n(x)dx\right],$$

$$\frac{d}{dx}\left[\lim_{n\doteq\infty} s_n(x)\right] = \lim_{n\doteq\infty}\left[\frac{d}{dx}s_n(x)\right].$$

The above questions then may be stated thus: Is the integral of the limit of the sum of an infinite number of quantities equal to the limit of the sum of the integrals of the quantities; and is it likewise in the case of the differentials?

For instance, given that $\quad \dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$

is $\quad \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)\left[i.e.\ \dfrac{1}{(1-x)^2}\right] = 1 + 2x + 3x^2 + \cdots ?$

and is $\quad \displaystyle\int_0^x \frac{dx}{1-x}\left[i.e.\ \log\frac{1}{1-x}\right] = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots ?$

169. Study of infinite series. Knowledge, elementary knowledge at least, of the theory of infinite series, and practice in their use are necessary in applied mathematics. Infinite series frequently present themselves in the theory and applications of the calculus, and accordingly the subject should be studied, to some extent at least, in an introductory course in calculus. The better text-books on algebra, for instance, among others, Chrystal's *Algebra* (Vol. II., Ed. 1889, Chap. XXVI., etc.), Hall and Knight's *Higher Algebra* (Chap. XXI.), contain discussions on infinite series and examples for practice.* Osgood's pamphlet, *Introduction to Infinite Series* (71 pages, Harvard University Publications), gives a simple, elementary, and excellent account of infinite series. "This pamphlet is designed to form a supplementary chapter on Infinite Series to accompany the text-book used in the course in calculus." *Recent text-books on the calculus*, in particular those of McMahon and Snyder, Lamb, and Gibson, contain definitions and theorems on infinite series; they will especially well repay consultation. More elaborate expositions of the properties of infinite series, which form parts of introductory courses in modern higher analysis, are given in Harkness and Morley, *Introduction to the Theory of Analytic Functions*, in particular

* Also see Hobson, *A Treatise on Plane Trigonometry*, Chap. XIV., and following chapters.

Chaps. VIII.–XI., and in Whittaker, *Modern Analysis*, in particular Chaps. II.–VIII. These discussions can be read, in large part, by one who possesses a knowledge of merely elementary mathematics.

A statement of a few of the principal definitions and theorems which are necessary for an elementary use of infinite series is given in Arts. 170–173.

170. Definitions. Algebraic properties of infinite series. An infinite series has been defined in Art. 167. If (see Art. 167) $\lim_{n \doteq \infty} s_n$ is a definite finite quantity, U say, the series is called **a convergent series**, and is said to converge to the value U. If s_n does not approach a definite finite value when n approaches infinity, the series is called **a divergent series.** In a divergent series, when n approaches infinity, s_n may either approach infinity, or remain finite but approach no definite value.

Thus, in Art. 167, series (1) is convergent; series (2) is convergent for values of x between -1 and $+1$, for then $s = \frac{1}{1-x}$; series (4) is convergent when r lies between -1 and $+1$, for then $s = \frac{a}{1-r}$. Series (5) is convergent for $p > 1$, and divergent for $p = 1$ and for $p < 1$. (Hall and Knight, *Algebra*, p. 235.)

[NOTE 1. *The harmonic series.* When $p = 1$, series (5) is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots.$$

This series is called *the harmonic series.*]

The series $1 + 2 + 3 + \cdots + n + \cdots$ is divergent. The series $1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + \cdots$, obtained by putting $x = 1$ in series (3), is divergent; for its limit is 0 or 1 according as n is even or odd. (A series that behaves like this is said to *oscillate.* Some writers do not include *oscillatory* series among the divergent series.)

In general only convergent series are regarded as of service in applied mathematics. (For the necessity of the qualifying phrase "in general," see Note 2.) A series may be employed to represent a function, or, what comes to the same thing, a function may be defined by a series, if the series is convergent. Thus series (2), Art. 167, may be used to represent or to define $\frac{1}{1-x}$, if x lies between -1 and $+1$. [See questions (a) and (b), Art. 168.*]

* Carl Friedrich Gauss (1777–1855), the great mathematician and astronomer of Göttingen, and Augustin-Louis Cauchy (1789–1857), professor at the

NOTE 2. **On divergent series.** Those who apply mathematics, astronomers in particular, have frequently obtained sufficiently good approximations to true results by means of divergent series. Such series, however, "cannot, except in special cases, and under special precautions, be employed in mathematical reasoning" (Chrystal, *Algebra*, Vol. II., p. 102). At the present time considerable attention is being paid by mathematicians to divergent series and to investigations of the fundamental operations of algebra and the calculus upon them. A work on the subject has recently appeared, viz. *Leçons sur les séries divergentes*, par Émile Borel (Paris, Gauthier-Villars, 1901, pp. vi + 182). "It is safe to say that no previous book upon divergent series has ever been written." Interesting and instructive information concerning divergent series will be found in reviews on this book, by G. B. Mathews (*Nature*, Nov. 7, 1901), and E. B. Van Vleck (*Science*, March 28, 1902).

Absolutely convergent series. A series the absolute values (see Art. 8, Note 1) of whose terms make a convergent series is said to be *absolutely* or *unconditionally convergent;* other convergent series are said to be **conditionally convergent.**

Ex. 1. Series (1), Art. 167, is an absolutely convergent series.

Ex. 2. The series $$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\cdots \qquad (a)$$
may be written $(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\cdots$, *i.e.* $\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\cdots$.

Series (a) may also be written
$$1-(\tfrac{1}{2}-\tfrac{1}{3})-(\tfrac{1}{4}-\tfrac{1}{5})\cdots, \textit{ i.e. } 1-\tfrac{1}{6}-\tfrac{1}{20}-\cdots.$$

Thus the value of the series (a), the terms being taken in the order indicated, is less than 1 and greater than $\frac{1}{2}$. It can also be shown that this series converges to a definite value. On the other hand (see Note 1, and the statement just preceding Note 1), the series
$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots$$
is divergent. Thus series (a) is a conditionally convergent series.

Theorems. (1) If a series is absolutely convergent, it is obvious that any series formed from it by changing the signs of any of the terms is also convergent.

Polytechnic School at Paris, who did much to make mathematics more rigorous than it had been during its rapid development in the eighteenth century, may be regarded as the founders of the modern theory of convergent series. James Gregory, professor of mathematics at Edinburgh, introduced the terms *convergent* and *divergent* in connection with infinite series in 1668.

(2) In a conditionally convergent series it is possible to rearrange the terms so that the new series will converge toward an arbitrary preassigned value.

(3) In an absolutely convergent series the terms can be rearranged at pleasure without altering the value of the series.

(4) If (see Art. 168) u and v are any two convergent series, they can be added term by term; that is, Equation (1), Art. 168, is true.

(5) If u and v are any two absolutely convergent series, they can be multiplied together like sums of a finite number of quantities; that is, Equation (2), Art. 168, is true.

For proofs and examples of these theorems see Osgood, *Introduction to Infinite Series*, Arts. 34, 35; Chrystal, *Algebra*, Vol. II., Chap. XXVI., §§ 12–14.

In a convergent series as n increases, s_n may either: (a) continually increase toward the limiting value of the series; or (b) decrease toward this limit; or (c) be alternately greater than and less than its limit.

Thus in series (1), Art. 167, s_n continually increases toward its limit (2); in the series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$, s_n is alternately greater than and less than its limit $\frac{2}{3}$.

Remainder after n terms. The symbol r_n or R_n is often used to denote the series (and also to denote the value of the series) formed by taking the terms after the nth, thus

$$r_n = u_n + u_{n+1} + u_{n+2} + \cdots.$$

This is usually called *the remainder after n terms.* Let a function be represented by a convergent series; *i.e.* let the value of the function be equivalent to the value of this convergent series. Then since

$$\text{the function} = \lim_{n \doteq \infty} s_n,$$

it follows that $$\lim_{n \doteq \infty} r_n = 0.$$

Interval of convergence. In general a convergent series, in a variable, x say, is convergent only for values of x in a certain interval, say from $x = a$ to $x = b$. The series is then said to converge within the interval (a, b), and this interval is called the *interval of convergence.*

Thus in series (2), Art. 167, the interval of convergence extends from $x = -1$ to $x = +1$. In this case, as in many others, the series is not convergent for the values of x (in this case -1 and $+1$) at the extremes of the interval. In some cases series are convergent for the values of the variable at the extremes of the interval of convergence as well as for the values between; in other cases a series may be convergent for the value of the variable at one extreme of the interval but not for the value at the other.

Power series. Series of the type

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \cdots,$$

in which the terms are arranged in ascending integral powers of x and the coefficients are independent of x, are called *power series* in x. A power series may converge for all values of x, but in general it will converge for some values of x and diverge for others.

Theorem. In the latter case the interval of convergence extends from some value $x = -r$ to the value $x = +r$; *i.e.* the value $x = 0$ is midway between the values of x at the extremes of the

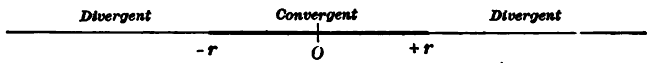

FIG. 101.

interval of convergence. Thus in the power series (2), Art. 167, the interval of convergence extends from -1 to $+1$. This theorem may be graphically represented, or illustrated, by Fig. 101.

(For proof of the theorem see Osgood, *Infinite Series*, Art. 18.)

171. Tests for convergence. Two simple tests for convergence will now be shown. For nearly all the infinite series occurring in elementary mathematics these tests will suffice to determine whether a series is convergent or divergent. These two tests are: (A) *the comparison test* and (B) *the test-ratio test.*

***A.* The comparison test.** Let there be two infinite series,

$$u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + \cdots, \tag{1}$$

and

$$v_0 + v_1 + v_2 + \cdots + v_{n-1} + v_n + \cdots. \tag{2}$$

If series (1) is convergent, and if each term of series (2) is not greater than the corresponding term of series (1) (*i.e.* if $v_n \leqq u_n$ for each value of n), then series (2) is convergent. If series (1)

is divergent, and if each term of series (2) is greater than the corresponding term of series (1), then series (2) is divergent.

Two series which are very useful for purposes of comparison are:

(*a*) The geometric series

$$a + ar + ar^2 + \cdots,$$

which is convergent when $|r| < 1$, divergent when $|r| \geqq 1$.

(*b*) The series $$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots,$$

which is convergent when $p > 1$, divergent when $p \leqq 1$ (see Art. 170).

Ex. 1. The series $$1 + \tfrac{1}{8} + \tfrac{1}{18} + \tfrac{1}{88} + \cdots$$

is convergent, for it is term by term not greater than the geometric convergent series $$1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} + \cdots.$$

***B*. The test-ratio test.** In series (6), Art. 167, the ratio

$$\frac{u_{n+1}}{u_n} \tag{3}$$

is commonly called the *test-ratio*. If when n increases beyond all bounds this ratio approaches a definite limit which is less than 1, then the series is convergent. For, suppose that ratio (3) is finite for all values of n, and suppose that after a certain finite number of terms, say m terms, it is less than a fixed number R which is less than 1. Now

$$s = u_1 + u_2 + \cdots + u_m + u_{m+1} + u_{m+2} + \cdots.$$

The sum of the first m terms is finite. Since

$$\frac{u_{m+1}}{u_m} < R, \quad \frac{u_{m+2}}{u_{m+1}} < R, \quad \cdots,$$

it follows that the series beginning with u_m is less than the geometric series $$u_m(1 + R + R^2 + \cdots),$$

and, accordingly, is less than

$$u_m \frac{1}{1 - R}.$$

Hence $$s < s_m + u_m \frac{1}{1-R},$$

and thus the series is convergent.

If when n increases beyond all bounds the test-ratio approaches a definite limit which is greater than 1, the series is divergent.

Ex. 2. Prove the last statement.

If the limiting value of the test-ratio is $+1$ or -1, further special investigation is necessary in order to determine whether the series is convergent or divergent.*

Thus the quality of the series, as regards its convergency or divergency, depends upon

$$\lim_{n \doteq \infty} \frac{u_{n+1}}{u_n}.$$

EXAMPLES.

3. Find whether the following series are convergent or divergent:

(1) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$, (2) $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots$,

(3) $1+\frac{1^2}{2!}+\frac{2^3}{3!}+\frac{3^4}{4!}+\frac{4^5}{5!}+\cdots$, (4) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\cdots$,

(5) $1+\frac{1}{2^p}+\frac{2}{3^p}+\frac{3}{4^p}+\frac{4}{5^p}\cdots$.

4. Examine the following series for convergency:

(1) $1+3x+5x^2+7x^3+9x^4+\cdots$, (2) $1^2+2^2x+3^2x^2+4^2x^3+5^2x^4+\cdots$,

(3) $1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$, (4) $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+\cdots$,

(5) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots+\frac{x^n}{n^2+1}+\cdots$.

---

* A series in which the absolute value of the test-ratio tends to the limit unity as n increases, will be absolutely convergent if, for all values of n after some fixed value,

$$\text{this absolute value} \leqq 1 - \frac{1+c}{n},$$

where c is a positive quantity independent of n. (For a proof of this general theorem, see Whittaker, *Modern Analysis*, Art. 13.)

172. Integration of infinite series. This article and the next will consider briefly the questions asked in Art. 168 (*e*). Principles (*a*) and (*b*) stated below are assumed in what follows. These principles and the theorem following are enunciated and proved in Osgood, *Infinite Series*, Arts. 37, 38, 39.

(*a*) A sufficient condition that the series of continuous functions

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$

represent a continuous function is given in the following theorem:

If $$u_0(x) + u_1(x) + \cdots, \qquad \alpha \leqq x \leqq \beta,$$

is a series of continuous functions convergent throughout the interval (α, β), *then the function* $f(x)$ *represented by this series will be continuous throughout this interval, if a set of positive numbers* M_0, M_1, M_2, $\cdots$, *independent of* x, *can be found such that*

(1) $|u_n(x)| \leqq M_n, \quad \alpha \leqq x \leqq \beta, \quad n = 0, 1, 2, \cdots;$

(2) $M_0 + M_1 + M_2 + \cdots$ is a convergent series.

(*b*) On applying condition (*a*), for a series to represent a continuous function, to power series it is found that:

A power series represents a continuous function within its interval of convergence. The function may, however, become discontinuous on the boundary of the interval.

Integration of infinite series term by term. Suppose that a continuous function $f(x)$ is represented throughout the interval (α, β) by an infinite series of continuous functions, thus,

$$f(x) = u_0(x) + u_1(x) + u_2(x) + \cdots, \tag{1}$$

this series accordingly being convergent for all values of x in the interval from $x = \alpha$ to $x = \beta$: it is required to determine the condition that

$$\int_\alpha^\beta f(x)dx = \int_\alpha^\beta u_0(x)dx + \int_\alpha^\beta u_1(x)dx + \int_\alpha^\beta u_2(x)dx + \cdots \tag{2}$$

be a true equation.

The second member of (2) is called the *term by term* integral of the series.

NOTE 1. Professor Osgood in an article, "A Geometrical Method for the Treatment of Uniform Convergence and Certain Double Limits" (*Bulletin of the Amer. Math. Soc.*, 2d series, Vol. III., Nov., 1896, page 62), gives an instance of a series which satisfies the conditions imposed on (1), but which cannot be integrated term by term.

On using the symbols $s_n(x)$ and $r_n(x)$ as defined in Arts. 167 and 170, Equation (1) can be written

$$f(x) = s_n(x) + r_n(x).$$

Accordingly,
$$\int_\alpha^\beta f(x)dx = \int_\alpha^\beta s_n(x)dx + \int_\alpha^\beta r_n(x)dx$$
$$= \int_\alpha^\beta u_0(x)dx + \int_\alpha^\beta u_1(x)dx + \cdots$$
$$+ \int_\alpha^\beta u_{n-1}(x)dx + \int_\alpha^\beta r_n(x)dx. \qquad (3)$$

Hence, *the necessary and sufficient condition that* (2) *be a true equation is that*
$$\lim_{n\doteq\infty} \int_\alpha^\beta r_n(x)dx = 0. \qquad (4)$$

It can be shown (see Osgood, *Infinite Series*, Art. 39) that this condition is satisfied by all series which satisfy the conditions of theorem (*a*). That is:

Series (1) *can always be integrated term by term,* i.e.

$$\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta u_0(x)\,dx + \int_\alpha^\beta u_1(x)\,dx + \int_\alpha^\beta u_2(x)\,dx + \cdots$$

will be a true equation, if a set of positive numbers, $M_0, M_1, M_2, \cdots$, *independent of* x, *can be found such that*

(1) $|u_n(x)| \leqq M_n, \quad \alpha \leqq x \leqq \beta, \quad n = 0, 1, 2, \cdots;$

(2) $M_0 + M_1 + M_2 + \cdots$ *is a convergent series.*

This test is "sufficiently general for most of the cases that arise in ordinary practice."

NOTE 2. In some works the discussion on integration of infinite series is prefaced by an explanation of what is called **uniform convergence,** and then it is shown that *uniformly convergent series can be integrated term by term.* (*E.g.* Gibson, *Calculus*, §§ 151, 155, which contain a highly commended general treatment of integration of infinite series, and which should be thoroughly studied by all who are interested in that topic; also see Lamb, *Calculus*, Arts. 194–196, where the integration of power series is discussed.) It is shown in § 7 of the article mentioned in Note 1 that the conditions imposed on the series in theorem (*a*) constitute a sufficient condition for the uniform convergence of that series.

Application to power series. *A power series can be integrated term by term throughout any interval* (α, β) *contained in the interval of convergence and not reaching out to the extremities of this interval.*

For proof of this theorem and for other applications of the preceding test, see Art. 40 of Osgood's pamphlet.

173. The differentiation of infinite series, term by term. *If the function* $f(x)$ *is represented by the series*

$$f(x) = u_0(x) + u_1(x) + \cdots \tag{1}$$

throughout the interval (α, β), *and if, moreover, the series of the derivatives*

$$u_0'(x) + u_1'(x) + \cdots$$

satisfy the conditions of Theorem (*a*), *Art.* 172, *throughout the interval* (α, β), *then will the derivative* $f'(x)$ *be given for any value of* x *in the interval by the series of derivatives;* i.e. *then will*

$$f'(x) = u_0'(x) + u_1'(x) + \cdots \tag{2}$$

be a true equation for all values of x *in the interval* (α, β).

Let the series of derivatives be denoted by $\phi(x)$; *i.e.* let

$$\phi(x) = u_0'x + u_1'(x) + \cdots. \tag{3}$$

It will now be shown that $\phi(x) = f'(x)$.

By Theorem (*a*), Art. 172, $\phi(x)$ is continuous, and the conditions, Art. 172, for the term by term integration of an infinite series are satisfied. Accordingly,

$$\begin{aligned}\int_a \phi(x)\,dx &= \int_a^x u_0'(x)\,dx + \int_a^x u_1'(x)\,dx + \cdots \\ &= [u_0(x) - u_0(\alpha) + u_1(x) - u_1(\alpha)] + \cdots \\ &= f(x) - f(\alpha).\end{aligned}$$

On differentiation, $\phi(x) = f'(x)$. Hence, Equation (2) is true.

By the aid of this theorem it can be proved that: **A power series** *can be differentiated term by term for any value of x within, but not necessarily for a value at, the extremities of the interval of convergence.* (For proof see Osgood, *Infinite Series,* p. 62.)

NOTE 1. For instances of functions defined by convergent series which cannot be differentiated term by term, see Professor Osgood's article mentioned in Note 1, Art. 172.

NOTE 2. Articles 172, 173, have been taken in large part from Osgood's *Introduction to Infinite Series,* § V., pp. 52–63. Also see Lamb, *Calculus,* Arts. 193–198; Gibson, *Calculus,* §§ 147–151, 155; the article mentioned in Note 1, Art. 172, §§ 5–9.

174. Applications of the integration and differentiation of series.

***A*. Expansions obtained by integration of known series.** Three important examples of the development of functions into infinite series by the aid of integration, will now be given.

EXAMPLES.

Ex. 1. For $-1 < x < 1$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots. \tag{1}$$

$$\therefore \int_0^x \frac{dx}{1+x^2} = \int_0^x dx - \int_0^x x^2\,dx + \int_0^x x^4\,dx - \cdots, \qquad \text{(Art. 172)}$$

i.e.
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots. \tag{2}$$

This is known as **Gregory's series.*** (For complete generality the term $\pm n\pi$, $(n=0, 1, 2, \cdots)$, should be in the second member.) Series (1) oscillates when $x = 1$; but by a theorem on series (see Chrystal, *Algebra,* Vol. II., Chap. XXVI., § 20) series (2) is convergent and represents $\tan^{-1} x$ even when $x = 1$.

NOTE 1. Series (2) can be used to calculate π. On putting $x = 1$ in (2), there is obtained

$$(a)\ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

* Discovered in 1670 by James Gregory (1638–1675), professor of mathematics at St. Andrews and later at Edinburgh. It was also found by Leibnitz (1646–1716). This series can also be derived independently of the calculus (see texts on Analytical Trigonometry).

This is a very slowly convergent series. More rapidly convergent series for calculating π are the following:

$$(b)\quad \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}; \qquad \text{(Machin's Series *)}$$

$$(c)\quad \frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}. \qquad \text{(Euler's Series †)}$$

EXERCISES. Show by elementary trigonometry that formulas (*b*) and (*c*) are true. Compute the value of π correctly to four places of decimals: (1) by using formula (*b*) and Gregory's series; (2) by using formula (*c*) and Gregory's series. (The correct value of π to ten places of decimals is 3.1415926536.)

Ex. 2. For $-1 < x < 1$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots.$$

On integrating between the end values 0 and 1, as in Ex. 1, there results

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{x^7}{7} + \cdots.$$

This series is due to Newton, and was used by him in computing the value of π. When $x = \frac{1}{2}$ this series gives

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2\cdot 3\cdot 2^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 2^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 2^7} + \cdots.$$

EXERCISE. Using the last result calculate π correctly to four places of decimals.

NOTE 2. For historical information concerning trigonometry and the computation of π, see Murray, *Plane Trigonometry*, Appendix, Note A, and Note C (Art. 6); Hobson, article "Trigonometry" (*Ency. Brit.*, 9th edition); also article "Squaring the Circle" (*Ency. Brit.*, 9th edition).

Ex. 3. For $-1 < x < 1$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots. \qquad (1)$$

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* John Machin, died 1751, was professor of astronomy at Gresham College, London.

† Leonhard Euler, 1707–1783.

On integrating between the end values 0 and x, as in Exs. 1, 2, there results

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots. \tag{2}$$

This is called **the logarithmic series.*** (*Here the base is e.*)

The members of (2) are equal for values of x as near 1 as one pleases. It is also easily shown that they are finite and continuous for $x = 1$. Accordingly, formula (2) is true also when $x = 1$.

On putting $x = 1$ in (2), $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, a very slowly convergent series.

On putting $x = -1$ in (2), $\log 0 = -(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots) = -\infty$. (See Art. 170.)

NOTE 3. Except for small values of x series (2) is very slowly convergent. A more rapidly convergent, and thus more useful, series for the computation of logarithms can be derived from (2), as follows. On putting $-x$ for x in (2),

$$\log(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots. \tag{3}$$

$$\therefore \log\frac{1+x}{1-x} = 2(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots). \tag{4}$$

On substituting $\dfrac{1}{2m+1}$ for x this becomes

$$\log\frac{m+1}{m} = 2\left[\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right]. \tag{5}$$

If $m = 1$, $\qquad \log 2 = 2\left(\dfrac{1}{3} + \dfrac{1}{3\cdot 3^3} + \dfrac{1}{5\cdot 3^5} + \cdots\right) = .693.$

If $m = 2$, $\log 3 - \log 2 = 2\left(\dfrac{1}{5} + \dfrac{1}{3\cdot 5^3} + \dfrac{1}{5\cdot 5^5} + \cdots\right) = .406.$

$$\therefore \log 3 = 1.099.$$

EXERCISES. (1) Find $\log 4$ to base e, by putting $m = 3$ in (5), assuming the value of $\log 3$. (2) Find the logarithms (to base e) of 5, 6, 7, 8, 9, 10, in a similar way. (The logarithms of 4, 5, 6, 7, 8, 9, 10, to base e, to three places of decimals, are respectively 1.386, 1.609, 1.792, 1.946, 2.079, 2.197, 2.303.)

***B.* Expansions obtained by the differentiation of known series.** Following is an example of the development of functions into infinite series by means of differentiation, the conditions of Art. 173 being satisfied.

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* Apparently first obtained in 1668 by Nicolaus Mercator of Holstein.

EXAMPLES.

4. When $-1 < x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots. \tag{1}$$

On differentiation,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

On differentiation and division by (2),

$$\frac{1}{(1-x)^3} = \frac{1}{2}(1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \cdots).$$

5. Show by successive differentiation of the members of Ex. 4 (1) that

$$\frac{1}{(1-x)^m} = (1-x)^{-m} = 1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots.$$

***C.* Approximate integration by means of series.** Under certain conditions (Art. 172) an approximate value of an integral may be obtained by means of an infinite series. Instances have been given in *A* above.

Thus, in Ex. 1, $\displaystyle\int_0^1 \frac{dx}{1+x^2} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots;$

in Ex. 2, $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots.$

EXAMPLES.

6. Find an approximate value of the length of the ellipse $x = a \sin\phi$, $y = b\cos\phi$. [Here ϕ is the complement of the eccentric angle for the point (x, y).]

It will be found (Art. 137) that

$$\text{length } s = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2\phi}\, d\phi. \tag{a}$$

On expanding the radical by the binomial theorem and taking the term by term integral of the resulting convergent series it will be found that

$$s = 2\pi a\left[1 - \left(\frac{1}{2}\right)^2\frac{e^2}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2\frac{e^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2\frac{e^6}{5} - \cdots\right]. \tag{b}$$

7. Apply result (*b*) of Ex. 6 to find the length of the ellipse whose semi-axes are 5 and 4. (To three places of decimals.)

8. The time of a complete oscillation of a simple pendulum of length l, oscillating through an angle $\alpha (< \pi)$ on each side of the vertical, is

$$4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \text{ in which } k=\sin\tfrac{1}{2}\alpha. \qquad (c)$$

Show that this time

$$=2\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2k^6+\cdots\right]. \qquad (d)$$

NOTE 4. Integrals (c) and (a) in Exs. 8 and 6 are known respectively as "elliptic integrals of the first and the second kind." The symbols $F(k, \phi)$, $E(e, \phi)$ are usually employed to denote these integrals (the upper end value here being ϕ). Knowledge of these integrals was specially advanced by Adrien Marie Legendre (1752–1833). See Art. 122, Note 4.

9. Show that:

(1) $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x^4}}=1+\frac{1}{2}\cdot\frac{1}{5}+\frac{1\cdot 3}{2\cdot 5}\cdot\frac{1}{9}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{13}+\cdots;$

(2) $\displaystyle\int_0^1\frac{dx}{\sqrt[3]{(1+x^3)^2}}=1-\frac{1}{4}\cdot\frac{2}{3}+\frac{1}{7}\cdot\frac{2\cdot 5}{1\cdot 2}\cdot\frac{1}{3^2}-\frac{1}{10}\cdot\frac{2\cdot 5\cdot 8}{1\cdot 2\cdot 3}\cdot\frac{1}{3^3}+\cdots;$

(3) $\displaystyle\int_0^1\frac{dx}{\sqrt[3]{1-x^5}}=1+\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{11}\cdot\frac{1\cdot 4}{1\cdot 2}\cdot\frac{1}{3^2}+\frac{1}{16}\cdot\frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{1}{3^3}+\cdots.$

CHAPTER XX.

TAYLOR'S THEOREM.

(See N.B. at beginning of Chapter XIX.)

175. Taylor's theorem is one of the most important theorems in the calculus. It has a wide application, and several important series, for example, the binomial series (see Ex. 6, Art. 176) can be derived by means of it. Let $f(x)$ be a function of x which is continuous throughout the interval from $x = a$ to $x = b$, and which also has all its derivatives continuous in this interval. Now let x receive an increment h. *Taylor's theorem* is a theorem which gives the development of the function $f(x + h)$ in a power series in h. The power series itself is called *Taylor's series.* (See Note 2, Art. 178.)

N.B. *In reading this chapter it is better to take up Art.* 180 *first.*

176. Derivation of Taylor's theorem. *Analytic proof of the theorem of mean value.* Let $f(x)$ and its first derivative be continuous in the interval from $x = a$ to $x = b$. Find R_1 so that

$$f(b) - f(a) = (b - a) R_1. \tag{1}$$

Substitute x for b, and put

$$F(x) = f(x) - f(a) - (x - a) R_1. \tag{2}$$

Then $F(b) = 0$ by (1); also $F(a) = 0$ identically. Hence, by Rolle's theorem (Art. 63),

$$F'(x_1) = 0,$$

in which x_1 lies between a and b. But by (2), on differentiation,

$$F'(x_1) = f'(x_1) - R_1.$$

Accordingly, $$R_1 = f'(x_1),$$

as already shown geometrically in Art. 64. Hence, from (1), on substituting x for b,

$$f(x) = f(a) + (x - a) f'(x_1), \tag{3}$$

in which $a \leqq x \leqq b$, and $a < x_1 < x$.

Result (3) may be written

$$f(x) = f(a) + (x - a) f'[a + \theta(x - a)], \tag{4}$$

in which $0 < \theta < 1$. Thus the theorem of mean value is deduced analytically from Rolle's theorem. (See Arts. 63, 64.)

Taylor's theorem. Taylor's theorem can be derived in various ways. The method adopted in this article is merely an extension of that used in deriving result (4).

Let $f(x)$ and its first n-derivatives be continuous in the interval from $x = a$ to $x = b$. Find R_n so that

$$f(b) - \left\{ f(a) + (b - a) f'(a) + \tfrac{1}{2}(b - a)^2 f''(a) + \cdots \right. \\ \left. + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) \right\} = (b - a)^n R_n. \tag{5}$$

Substitute x for a, and let

$$F(x) = f(b) - f(x) - (b - x) f'(x) - \tfrac{1}{2}(b - x)^2 f''(x) - \cdots \\ - \frac{(b - x)^{n-1}}{(n-1)!} f^{(n-1)}(x) - (b - x)^n R_n. \tag{6}$$

Then $F(x)$ is continuous in the interval from $x = a$ to $x = b$, since $f(x)$ and its first n-derivatives are continuous there. By (5), $F(a) = 0$; also $F(b) = 0$ identically. Hence, by Rolle's theorem,

$$F'(x_1) = 0,$$

in which $a < x_1 < b$.

But, on differentiation, reduction, and substitution, in (6),

$$F'(x_1) = -\frac{(b - x_1)^{n-1}}{(n-1)!} f^n(x_1) + n(b - x_1)^{n-1} R_n. \tag{7}$$

$$\therefore R_n = \frac{1}{n!} f^n(x_1) = \frac{1}{n!} f^n[a + \theta(x - a)], \tag{8}$$

in which $0 < \theta < 1$.

On substituting this value of R_n in (5), and writing x for a and $x+h$ for b, there is obtained

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(x)$$
$$+ \frac{h^n}{n!}f^{(n)}(x+\theta h). \qquad (9)$$

This is **Taylor's theorem with the remainder,** the last term of the second member being denoted as **the remainder.** In formula (9) x and $x+h$ must both be in the interval of continuity; in any particular application of this formula, x has a fixed value and h varies. Theorem [or formula] (9) is true for all functions which, with their first n-derivatives, are continuous in the assigned interval of continuity. *If all the derivatives of $f(x)$ are continuous in the interval, and if*

$$\lim_{n \doteq \infty} \frac{h^n}{n!}f^{(n)}(x+\theta h) = 0,$$

then $$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots. \qquad (10)$$

For (by Art. 170) the infinite series in the second member converges to the value of $f(x+h)$ and, accordingly, represents the function $f(x+h)$. Formula (10) is called **Taylor's theorem,** and the series is called **Taylor's series.** In (9) and (10) h may be positive or negative, so long as x and $x+h$ are in the interval of continuity. "*The remainder,*" the last term in (9), represents the limit of the sum of all the terms after the nth term of the infinite series in (10); it is the amount of the error that is made when the sum of the first n-terms of the series is taken as the value of the function.

NOTE. The above method of proving the theorem of mean value was first given by Joseph Alfred Serret (1819–1885), professor of the Sorbonne in Paris, in his *Cours de calcul différentiel et intégral*, 2e éd., t. I., page 17 *seq.* The above proof of Taylor's theorem appears in Harnack's *Calculus* (Cathcart's translation, Williams and Norgate), pages 65, 66, and in Gibson's *Calculus*, pages 390–393. The proof in Echols's *Calculus* (p. 82) is likewise based on the theorem of mean value.

Taylor's theorem and series are important in the theory of functions of a complex variable, and are more fully investigated in that subject.

EXAMPLES.

1. Express $\log(x+h)$ by an infinite series in ascending powers of h.

Here
$$f(x+h) = \log(x+h).$$
$$\therefore f(x) = \log x,$$
$$f'(x) = \frac{1}{x},$$
$$f''(x) = -\frac{1}{x^2},$$
$$f'''(x) = \frac{2}{x^3}, \text{ etc.}$$
$$\therefore \log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \cdots.$$

Here x must not be 0, for then $f(x) = -\infty$, and thus is discontinuous for $x = 0$. The series is evidently more rapidly convergent the smaller is h and the larger is x.

On putting $x = 1$ and $h = 1$, this result gives
$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots,$$
as found in Ex. 3, Art. 174.

If the finite series in (9) is used, then
$$\log(x+h) = \log x + \frac{h}{x} + \frac{h^2}{2x^2} + \cdots + (-1)^{n-1}\frac{h^n}{n!\,(x+\theta h)^n},\ 0<\theta<1.$$
Here, if $x = h = 1$,
$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + (-1)^{n-1}\cdot\frac{1}{n(1+\theta)^n}.$$

On interchanging h and x in formula (10), if that can be done in the interval of continuity, there is obtained the following form of Taylor's theorem:
$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots, \qquad (11)$$
a form which is often useful. Similarly in the case of formula (9).

2. Express $\log(x+h)$ by an infinite series in ascending powers of x.

Here $f(x+h) = \log(x+h)$. $\therefore f(h) = \log h,\ f'(h) = \frac{1}{h},\ f''(h) = -\frac{1}{h^2}$, etc.
$$\therefore \log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \cdots.$$

If $h = 1$, $\qquad \log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots,$

as already obtained in Ex. 3, Art. 174.

3. Represent $\sin(x+h)$ by an infinite series in ascending powers in h.

Here $f(x+h)=\sin(x+h)$. $\therefore f(x)=\sin x$, $f'(x)=\cos x$, $f''(x)=-\sin x$, etc.

Hence, on using formula (10),

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{2!}\sin x-\frac{h^3}{3!}\cos x+\frac{h^4}{4!}\sin x+\cdots.$$

Let $x=\frac{\pi}{3}$, and $h=\frac{1}{100}$ of a radian (*i.e.* 34′ 22″.65).

Then

$$\sin\left(\frac{\pi}{3}+\frac{1}{100}\right)^r=\sin\frac{\pi}{3}+\frac{1}{100}\cos\frac{\pi}{3}-\frac{1}{(100)^2 2!}\sin\frac{\pi}{3}-\frac{1}{(100)^3 3!}\cos\frac{\pi}{3}+\cdots.$$

This is a rapidly convergent series.

Now $\sin\frac{\pi}{3}=.86603$, $\cos\frac{\pi}{3}=.50000$. On making the computations, it will be found that, to *five* places of decimals, $\sin 60° 34' 22''.65=.87099$.

Note. The last exercise is an example of one of the most useful practical **applications of Taylor's theorem.** Namely, *if a value of a function is known for a particular value of the variable, then the value of the function for a slightly different value of the variable can be computed from the known value by Taylor's formula.* (See Art. 27, Notes 1, 3; Art. 82, Note 3.)

4. Expand $\sin(x+h)$ in a series in ascending powers of x.

In this case form (11) is to be used. Here $f(x+h)=\sin(x+h)$. $\therefore f(h)=\sin h$, $f'(h)=\cos h$, $f''(h)=-\sin h$, $f'''(h)=-\cos h$, etc.

$$\therefore \sin(x+h)=\sin h+x\cos h-\frac{x^2}{2!}\sin h-\frac{x^3}{3!}\cos h+\cdots.$$

On letting $h=0$, the following important series is obtained:

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots.$$

5. Expand $\cos(x+h)$ in series, (a) in ascending powers of h, (b) in ascending powers of x. From the latter form deduce the series

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots.$$

6. Expand $(x+h)^m$ by Taylor's formula in a power series in h, and thus obtain the Binomial Expansion

$$(x+h)^m=x^m+mx^{m-1}h+\frac{m\cdot m-1}{1\cdot 2}x^{m-2}h^2+\cdots.$$

(This series is convergent for $h<1$, divergent for $h>1$. The case in which $h=\pm 1$ requires special investigation.)

7. Given that $f(x)=4x^3-3x^2+7x+5$, develop $f(x+2)$ and $f(x-3)$ by Taylor's expansion. Then find $f(x+2)$ and $f(x-3)$ by the usual algebraic method, and thus verify the results.

8. (1) Assuming $\sin 42°$, compute $\sin 44°$ and $\sin 47°$ by Taylor's expansion. (2) Assuming $\cos 32°$, compute $\cos 34°$ and $\cos 37°$ by Taylor's expansion. (3) Do further exercises like (1) and (2).

9. Derive $\log(x+h)=\log h+\frac{x}{h}-\frac{x^2}{2h^2}+\frac{x^3}{3h^3}-\frac{x^4}{4h^4}+\cdots$, when $|x|<1$; $\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\cdots$, when $|x|>1$.

10. Show that

$$\log\sin(x+a)=\log\sin x+a\cot x-\frac{a^2}{2}\csc^2 x+\frac{a^3}{3}\frac{\cos x}{\sin^3 x}+\cdots.$$

177. Another form of Taylor's theorem. On substituting the value of R_n [Eq. (8), Art. 176] in (5) and writing x for b, there is obtained

$$f(x)=f(a)+(x-a)f'(a)+\tfrac{1}{2}(x-a)^2f''(a)+\cdots+\frac{(x-a)^{n-1}}{\lfloor n-1}f^{(n-1)}(a)$$
$$+\frac{(x-a)^n}{n!}f^{(n)}[a+\theta(x-a)]. \quad (1)$$

If all the derivatives of $f(x)$ are continuous in the assigned interval, and

$$\lim_{n\doteq\infty}\frac{(x-a)^n}{n!}f^{(n)}[a+\theta(x-a)]=0,$$

then (Art. 170) the infinite series $f(a)+(x-a)f'(a)+\frac{1}{2}(x-a)^2f''(a)+\cdots$ represents the function $f(x)$ *; *i.e.*

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots$$
$$+\frac{(x-a)^n}{n!}f^{(n)}(a)+\cdots. \quad (2)$$

Forms (1) and (2) for Taylor's theorem and series, are frequently useful. The last term in the finite series (1) is *Lagrange's form of the remainder in Taylor's series.* (See Note 4, Art. 178.)

* Except in some rare cases.

EXAMPLES.

1. Express $5x^2 + 7x + 3$ in powers of $x - 2$.

Here

$$\begin{aligned} f(x) &= 5x^2 + 7x + 3, & \therefore f(2) &= 37, \\ f'(x) &= 10x + 7, & f'(2) &= 27, \\ f''(x) &= 10, & f''(2) &= 10, \\ f'''(x) &= 0, & f'''(2) &= 0. \end{aligned}$$

Now by (2), $f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2}f''(2) + \cdots$.

$$\therefore 5x^2 + 7x + 3 = 37 + 27(x-2) + 5(x-2)^2.$$

2. Express $4x^3 - 17x^2 + 11x + 2$ in powers of $x + 3$, in powers of $x - 5$, and in powers of $x - 4$, and verify the results.

3. Express $5y^4 + 6y^3 - 17y^2 + 18y - 20$ in powers of $y - 4$ and in powers of $y + 4$, and verify the results.

NOTE. Exs. 1–3 can be solved, perhaps more rapidly, by *Horner's process.* (See text-books on algebra, *e.g.* Hall and Knight's *Algebra*, § 549, 4th edition, 1889.)

4. Develop e^x in powers of $x - 1$.

5. Show that $\frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \cdots$, when x varies from $x = 0$ to $x = 2a$.

6. Show that $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$ is true for values of x between 0 and 2.

178. Maclaurin's theorem and series. This is a theorem for expanding a function in a power series in x. As will be seen presently, it is really a special case of Taylor's theorem.

Let $f(x)$ and its first n derivatives be finite for $x = 0$ and be continuous for values of x in the neighborhood of $x = 0$.

In form (9), Art. 176, put $x = 0$; then

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{h^n}{n!}f^{(n)}(\theta h).$$

On writing x for h, this becomes

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x). \quad (1)$$

If $f(x)$ and *all* its derivatives are finite for $x = 0$, and if

$$\lim_{n \doteq \infty} \frac{x^n}{n!} f^{(n)}\theta(x) = 0, \quad \text{then}$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots. \qquad (2)$$

This is known as **Maclaurin's theorem**, and the series is called **Maclaurin's series.** The last term in (1) is called *the remainder in Maclaurin's series.* It is the limit of the sum of the terms of the series after the nth term.

EXAMPLES.

1. Show that formula (2) comes from form (11), Art. 176, on putting $h = 0$; show that this has practically been done in the derivation above. Show that formula (2) comes from form (2), Art. 177, on putting $a = 0$.

2. Develop $\sin x$ in a power series in x.

$$\begin{array}{lll} \text{Here} & f(x) = \sin x. & \therefore f(0) = 0, \\ & \therefore f'(x) = \cos x, & f'(0) = 1, \\ & f''(x) = -\sin x, & f''(0) = 0, \\ & f'''(x) = -\cos x, & f'''(0) = -1, \\ & f^{\text{iv}}(x) = \sin x, & f^{\text{iv}}(0) = 0, \\ & \text{etc.} & \text{etc.} \end{array}$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + \cdots. \qquad (A)$$

(Compare Ex. 2 above and Ex. 4, Art. 176.)

On applying the method of Art. 171 it will be found that the interval of convergence is from $-\infty$ to $+\infty$.

3. Calculate $\sin(\frac{1}{10}$ radian), *i.e.* $\sin 5° 43' 46''.5$.

By A, $\sin(.1 \text{ radian}) = .1 - \frac{(.1)^3}{3!} + \frac{(.1)^5}{5!} - \cdots = .09983.$

4. Calculate $\sin(.5^r)$ and $\sin(.2^r)$ to 5 places of decimals. (For results, see Trigonometric Tables.)

5. Show that $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad (B)$$

and show that the interval of convergence is from $-\infty$ to $+\infty$.

6. To 4 places of decimals calculate the following: $\sin(.3^r)$, $\cos(.2)^r$, $\sin(.4^r)$, $\cos(.4^r)$. (See values in Trigonometric Tables.)

7. Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, (C)

and show that this series is convergent for every finite value of x.

8. Substitute 1 for x in C, and thus deduce 2.71828 as an approximate value of e.

9. Assuming A and B deduce that the sine of the angle of magnitude zero, is zero, and that the cosine of this angle is unity.

NOTE 1. Expansions A and B were first given by Newton in 1669. He also first established series C. These expansions can also be obtained by the ordinary methods of algebra, without the aid of the calculus. For this derivation see Chrystal, *Algebra*, Part II., Chap. XXIX., § 14, Chap. XXVIII., § 5, and the texts of Colenso, Hobson, Locke, Loney, and others, on what is frequently termed *Analytical Trigonometry*, or *Higher Trigonometry*. [This subject is rather to be regarded as a part of algebra (Chrystal, *Algebra*, Part II., p. vii).] Also see article "Trigonometry" (*Ency. Brit.*, 9th ed.).

10. Develop the following functions in ascending powers in x: (1) $\sec x$; (2) $\log \sec x$; (3) $\log(1+x)$. Compare the latter with Ex. 3, Art. 174.

11. Show that $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$.
By this series compute $\tan(.5^r)$, $\tan 15°$, $\tan 25°$.

12. Find: (1) $\int e^x \cos x\, dx$; (2) $\int_a^b \frac{e^x}{x} dx$; (3) $\int_0^x e^{-x^2} dx$.

NOTE 1 *a*. The integral in Ex. 12 (3) is important in the theory of probabilities. If the end-value x is ∞, the value of the integral is $\frac{1}{2}\sqrt{\pi}$. (Williamson, *Integral Calculus*, Ex. 4, Art. 116.)

13. Assuming the series for $\sin x$, prove Huyhen's rule for calculating approximately the length of a circular arc, viz.: From eight times the chord of half the arc subtract the chord of the whole arc, and divide the result by three.

14. State Maclaurin's theorem, and from the expansion for $\tan x$ find the value of $\tan x$ to three places of decimals when $x = 10°$.

15. Show that $\cos^n x = 1 - \frac{n}{2!}x^2 + \frac{n(3n-2)}{4!}x^4 - \cdots$.

Note 2. Historical. *Taylor's theorem*, or formula, was discovered by Dr. Brook Taylor (1685–1731), an English jurist, and published in his *Methodus Incrementorum* in 1715. It was given as a corollary from a theorem in *Finite Differences*, and appeared without qualifications, there being no reference to a remainder. The formula remained almost unnoticed until Lagrange (1736–1813) discovered its great value, investigated it, and found for the

remainder the expression called by his name. His investigation was published in the *Mémoires de l'Académie de Sciences à Berlin* in 1772. "Since then it has been regarded as the most important formula in the calculus."

Maclaurin's formula was named after Colin Maclaurin (1698–1746), professor of mathematics at Aberdeen 1718 ?–1725, and at Edinburgh, 1725–1745, who published it in his *Treatise on Fluxions* in 1742. It should rather be called *Stirling's* theorem, after James Stirling (1690–1772), who first announced it in 1717 and published it in his *Methodus Differentialis* in 1730. Maclaurin recognized it as a special case of Taylor's theorem, and stated that it was known to Stirling; Stirling also credits it to Taylor.

NOTE 3. Taylor's and Maclaurin's theorems are virtually identical. It has been shown in Art. 178 that Maclaurin's formula can be deduced from Taylor's. On the other hand, Taylor's formula can be deduced from Maclaurin's; *e.g.* see Lamb's *Calculus*, page 567, and Edwards's *Treatise on Differential Calculus*, page 81.

NOTE 4. *Forms of the remainder for Taylor's series* (2), Art. (177). *Lagrange's form of the remainder* has already been noticed in Art. 177. Another form, viz.

$$\frac{(x-a)^n(1-\theta)^{n-1}}{(n-1)\,!}f^{(n)}[a+\theta(x-a)],\ 0<\theta<1,$$

was found by Cauchy (1789–1857), and first published in his *Leçons sur le Calcul infinitesimal* in 1826. A more general form of the remainder is the *Schlömilch-Roche* form, devised subsequently, viz.

$$\frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)\,!\,p}f^{(n)}[a+\theta(x-a)],\ 0<\theta<1.$$

This includes the forms of Lagrange and Cauchy; for these forms are obtained on substituting n and 1 respectively for p. (The θ's in these forms are not the same, but are alike in being numbers between 0 and 1.) In particular expansions some one of these forms may be better than the others for investigating the series after the first n terms.

NOTE 5. **Extension of Taylor's theorem to functions of two or more variables.** For discussions on this topic see McMahon and Snyder's *Calculus*, Art. 103; Lamb's *Calculus*, Art. 211; Gibson's *Calculus*, § 157.

NOTE 6. **References for collateral reading** *on Taylor's theorem.* Lamb, *Calculus*, Chap. XIV.; McMahon and Snyder, *Diff. Cal.*, Chap. IV.; Gibson, *Calculus*, Chaps. XVIII., XIX.; Echols, *Calculus*, Chap. VI.

179. Relations between trigonometric (or circular) functions and exponential functions. The following important relations, which are extremely useful and frequently applied, can be deduced from the expansions for $\sin x$, $\cos x$, and e^x in Art. 178.

The substitution of ix for x in C gives

$$e^{ix} = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots + i\left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots\right) = \cos x + i \sin x. \quad (1)$$

The substitution of $-ix$ for x in C gives

$$e^{-ix} = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots - i\left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots\right) = \cos x - i \sin x. \quad (2)$$

From (1) and (2), on addition and subtraction,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad (3), \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \quad (4)$$

On putting π for x in (1), there is obtained the striking relation

$$e^{i\pi} = -1. \qquad \text{(See Art. 38, Note on } e.\text{)}$$

NOTE 1. The remarkable relations (1)–(4), by which the sine and cosine of an angle can be expressed in terms of certain exponential functions of the angle (measured in *radians*), and conversely, were first given by Euler (1707–1783). (In connection with the expansions in Arts. 178, 179, see the historical sketch in Murray's *Plane Trigonometry*, Appendix, Note A; in particular pp. 168, 169.)

NOTE 2. Results (1)–(4) can also be deduced by the methods of ordinary algebra; see Note 1, Art. 178, the references therein, and Chrystal's *Algebra*, Part II., Chap. XXIX., § 23.

EXAMPLES.

1. From (3) and (4) deduce that $\cos^2 x + \sin^2 x = 1$.

2. Show that $\tan x = \dfrac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}$.

3. Express $\cot x$, $\sec x$, $\operatorname{cosec} x$, in terms of exponential functions of x.

NOTE 3. Since, by (1), $e^{i\phi} = \cos\phi + i\sin\phi$, and $e^{in\phi} = \cos n\phi + i \sin n\phi$, and since $(e^{i\phi})^n = e^{in\phi}$, it is evident that

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi,$$

for all values of n, positive or negative, integral or fractional.

This very important theorem is called *De Moivre's theorem*, after its discoverer Abraham de Moivre (1667–1754), a French mathematician who settled in England. It first appeared in his *Miscellanea Analytica* (London, 1730), a work in which "he created 'imaginary trigonometry.'" [On *De Moivre's theorem*, and results (1)–(4), see Murray, *Plane Trigonometry*, Art. 98, and Appendix, Note D; and other text-books on Trigonometry.]

N.B. The article on **Hyperbolic Functions**, Appendix, Note A, may be conveniently read at this time.

180. Another method of deriving Taylor's and Maclaurin's series. Following is a method which is more generally employed than that in Arts. 176 and 178 for finding the forms of the series of Taylor and Maclaurin.

***A*. Maclaurin's series.** Let $f(x)$ and its derivatives be continuous in the neighbourhood of $x = 0$, say from $x = -a$ to $x = a$. Suppose that $f(x)$ can be expressed in a power series in x convergent in the interval $-a$ to $+a$. That is, assume that (for $-a < x < a$) there can be an identically true equation of the form

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots + A_nx^n + \cdots. \tag{1}$$

The coefficients $A_0, A_1, A_2, \cdots, A_n, \cdots$, will now be found. It has been seen in Art. 173 that if Equation (1) is identically true, then the equation obtained by differentiating both members of (1), viz.

$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + \cdots + nA_nx^{n-1} + \cdots,$$

also is identically true for values of x in some interval that includes zero. For the same reason the following equations, obtained by successive differentiation, are also identical in intervals that include zero, viz.:

$$f''(x) = 2A_2 + 2 \cdot 3A_3x + \cdots + n(n-1)A_nx^{n-2} + \cdots,$$
$$f'''(x) = 2 \cdot 3 \cdot A_3 + \cdots + n(n-1)(n-2)A_nx^{n-3} + \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f^{(n)}(x) = n \cdot n-1 \cdot n-2 \cdot \cdots \cdot 2 \cdot 1\, A_n + \cdots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

On putting $x = 0$ in each of these identities it is found that

$$A_0 = f(0),\ A_1 = f'(0),\ A_2 = \frac{f''(0)}{2!},\ A_3 = \frac{f'''(0)}{3!},\ \cdots,\ A_n = \frac{f^{(n)}(0)}{n!},\ \cdots.$$

Hence, on substitution in (1),

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots, \tag{2}$$

which is Maclaurin's series (Art. 178).

***B*. Taylor's series.** Let $f(x)$ and its derivatives be continuous in the neighbourhood of $x = a$, say from $x = a - h$ to $x = a + h$. Suppose that $f(x)$ can be expressed in a power series in $x - a$

which is convergent in the neighbourhood of $x = a$. In other words, suppose that there is an identically true equation of the form

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \cdots + A_n(x-a)^n + \cdots. \tag{3}$$

Then, as in case A, the following equations, which are obtained by successive differentiation, also are identically true for values of x near $x = a$, viz.:

$$f'(x) = A_1 + 2\,A_2(x-a) + 3\,A_3(x-a)^2 + \cdots + nA_n(x-a)^{n-1} + \cdots,$$

$$f''(x) = 2\,A_2 + 2\cdot 3\,A_3(x-a) + \cdots + n\cdot n-1\cdot A_n(x-a)^{n-2} + \cdots,$$

$$f'''(x) = 2\cdot 3\cdot A_3 + \cdots + n\cdot n-1\cdot n-2\cdot A_n(x-a)^{n-3} + \cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^n(x) = n\cdot n-1\cdot n-2\cdot\cdots 2\cdot 1\cdot A_n + \cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

On putting $x = a$ in each of these identities it is found that

$$A_0 = f(a),\ A_1 = f'(a),\ A_2 = \frac{f''(a)}{2!},\ A_3 = \frac{f'''(a)}{3!},\ \cdots,$$

$$A_n = \frac{f^{(n)}(a)}{n!},\ \cdots.$$

Hence, on substitution in (3),

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots, \tag{4}$$

which is series (2), Art. 177.

If in (4) x is changed into $x + a$, then

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \cdots + \frac{x^n}{n!}f^{(n)}(a) + \cdots, \tag{5}$$

which is series (11), Art. 176, with a written for h. On interchanging a and x in (5), form (10), Art. 176, is obtained.

Note. On the proof of Taylor's theorem. The above merely shows the derivation of the *form* of Taylor's series. It is still necessary to examine into the convergency or divergency of the series and to determine the remainder

after any number of terms. The investigation of the validity of the series is a very important matter in the calculus. For this investigation see, among other works, Todhunter, *Diff. Cal.*, Chap. VI.; Williamson, *Diff. Cal.*, Arts. 73–77; Edwards, Treatise on *Diff. Cal.*, Arts. 130–142; McMahon and Snyder, *Diff. Cal.*, Chap. IV.; Lamb, *Calculus*, Arts. 203, 204; article, "Infinitesimal Calculus" (*Ency. Brit.*, 9th ed., §§ 46–52).

181. Application of Taylor's theorem to the determination of conditions for maxima and minima. This article is supplementary to Art. 76. Let $f(x)$ be a function of x such that $f(a+h)$ and $f(a-h)$ can be developed in Taylor's series; and let it be required to determine whether $f(a)$ is a maximum or minimum value of $f(x)$. On developing $f(a-h)$ and $f(a+h)$ by formula (9), Art. 176,

$$f(a-h)=f(a)-hf'(a)+\frac{h^2}{2!}f''(a)-\frac{h^3}{3!}f'''(a)+\cdots$$
$$+\frac{(-h)^n}{n!}f^{(n)}(a-\theta_1 h), \tag{1}$$

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots$$
$$+\frac{h^n}{n!}f^{(n)}(a+\theta_2 h), \tag{2}$$

in which θ_1 and θ_2 lie between 0 and 1.

Suppose that the first $n-1$ derivatives of $f(x)$ are zero when $x=a$, and that the nth derivative does not vanish for $x=a$. Then

$$f(a-h)-f(a)=\frac{(-h)^n}{n!}f^{(n)}(a-\theta_1 h), \tag{3}$$

$$f(a+h)-f(a)=\frac{h^n}{n!}f^{(n)}(a+\theta_2 h). \tag{4}$$

It follows from the hypothesis concerning $f(x)$ that the signs of $f^{(n)}(a-\theta_1 h)$ and $f^{(n)}(a+\theta_2 h)$, for infinitesimal values of h, are the same as the sign of $f^{(n)}(a)$. From (3), (4), and the definitions of maxima and minima, it is obvious that:

(*a*) *If n is odd,* the first members of (3) and (4) have opposite signs, and consequently, $f(a)$ *is neither a maximum nor a minimum value of* $f(x)$; (*b*) *If n is even and $f^{(n)}(a)$ is positive,* the first members of (3) and (4) are both positive, and consequently, $f(a)$ *is a*

minimum value of $f(x)$; (*c*) *If* n *is even and* $f^{(n)}(a)$ *is negative*, the first members of (3) and (4) are both negative, and consequently, $f(a)$ *is a maximum value of* $f(x)$. The condition for maxima and minima that was deduced in Art. 76, (*c*), is a special case of this, viz. the case in which $n=2$.

182. Application of Taylor's theorem to the deduction of a theorem on contact of curves. This article is supplementary to Art. 143. (See Art. 143, Note 4.)

Theorem. *If two curves have contact of an even order, they cross each other at the point of contact; if two curves have contact of an odd order, they do not cross each other at the point of contact.*

Let the two curves $y=\phi(x)$ and $y=\psi(x)$ (1)

have contact of the nth order at $x=a$. Then

$$\phi(a)=\psi(a),\ \phi'(a)=\psi'(a),\ \phi''(a)=\psi''(a),\ \cdots,\ \phi^{(n)}(a)=\psi^{(n)}(a). \quad (2)$$

Now compare the ordinates of these curves at $x=a-h$, *i.e.* compare $\phi(a-h)$ and $\psi(a-h)$; also compare the ordinates at $x=a+h$, *i.e.* compare $\phi(a+h)$ and $\psi(a+h)$. Let it be further premised that $\phi(a\pm h)$ and $\psi(a\pm h)$ can be expanded in Taylor's series. On using Taylor's theorem (form 9, Art. 176), and remembering hypothesis (2), it will be found that

$$\phi(a-h)-\psi(a-h)=\frac{(-h)^{n+1}}{(n+1)!}[\phi^{(n+1)}(a-\theta_1 h)-\psi^{(n+1)}(a-\theta_2 h)], \quad (3)$$

$$\phi(a+h)-\psi(a+h)=\frac{h^{n+1}}{(n+1)!}[\phi^{(n+1)}(a-\theta_3 h)-\psi^{(n+1)}(a-\theta_4 h)], \quad (4)$$

in which the four θ's all lie between 0 and 1.

Let h approach zero; then, by the premise above, the signs of the expressions in brackets are the same as the signs of $[\phi^{(n+1)}(a)-\psi^{(n+1)}(a)]$. Hence, *if n is odd*, the first members of (3) and (4) have the same sign, and, accordingly, *the curves do not cross; if n is even*, these first members have opposite signs, and, accordingly, *the curves do cross.*

Ex. Accompany the proof of this theorem with illustrative figures.

183. Applications of Taylor's theorem in elementary algebra. Let $f(x)$ be a rational integral function of x, of the nth degree say. Then $f^{(n+1)}(x)$ and the following derivatives are all zero. Hence, Taylor's series for $f(x+h)$ in ascending powers of either h or x [see forms (10) and (11), Art. 176] is *finite*. That is,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots+\frac{h^n}{n!}f^{(n)}(x), \tag{1}$$

$$f(x+h)=f(h)+xf'(h)+\frac{h^2}{2!}f''(h)+\cdots+\frac{x^n}{n!}f^{(n)}(h). \tag{2}$$

A rational integral function $f(x)$ of the nth degree can also be expressed in a *finite* series in ascending powers of $x-a$ [see form (2), Art. 177]. That is,

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^{(n)}(a). \tag{3}$$

EXERCISE. See Ex. 7, Art. 176, and Exs. 1, 2, 3, Art. 177.

NOTE 1. Let $f(x)$ be as specified above. In general the calculation of $f(x+h)$ and the expression of $f(x)$ in terms of $x-a$, can be more speedily effected by *Horner's process*.* This process is shown in various texts on algebra; *e.g.* Hall and Knight's *Algebra* (4th edition), Arts. 549, 572.

NOTE 2. For an **application of Taylor's theorem to interpolation,** see McMahon and Snyder, *Calculus*, Note, pp. 325, 326.

NOTE 3. In expansion (10), Art. 176, if h is a differential dx of x, then h, h^2, h^3, $\cdots$, are respectively differentials of x of the first, second, third, $\cdots$, orders; and $hf(x)$, $h^2f''(x)$, $h^3f'''(x)$, $\cdots$, are respectively differentials of $f(x)$ of the first, second, third, $\cdots$, orders. If h (or dx) is an infinitesimal, these differentials are also infinitesimals of the respective orders mentioned.

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* William George Horner (1786–1837), an English mathematician, who discovered a very important method of finding approximate solutions of numerical equations of any degree.

CHAPTER XXI.

DIFFERENTIAL EQUATIONS.

N.B. The references made in this chapter are to Murray, *Differential Equations.*

184. Definitions. Classifications. Solutions. This chapter is concerned with showing how to obtain solutions of a few differential equations which the student is likely to meet in elementary work in mechanics and physics.

Differential equations are equations that involve derivatives or differentials. Such equations have often appeared in the preceding part of this book.

Thus, in Art. 37, Exs. 2, 11, 13, differential equations appear; Equations (1), Art. 60, (2)–(5), Art. 67 (*a*), (2)–(5), Art. 67 (*c*), (3)–(6), Art. 67 (*d*), are differential equations; so also, in Art. 68, are (1) and (2), Ex. 5; equations in Exs. 13, 14, and some of the equations in Exs. 10, 11; several equations in Ex. 1, Art. 69; Equations (2)–(4), Ex. 1, Art. 73; the answers to Exs. 2–4, Art. 73; in Ex. 4, Art. 79; in Exs. 5–8, Art. 80; Equation (8), Art. 144; etc., etc.

Differential equations are classified in the following ways, *A* and *B*:

A. Differential equations are classified as **ordinary** *differential equations* and **partial** *differential equations,* according as one, or more than one, independent variable is involved. Thus, the equations in Ex. 4, Art. 79, and in Exs. 5–8, Art. 80, are partial differential equations; the other equations mentioned above are ordinary differential equations. (Only ordinary differential equations are discussed in this chapter.)

B. Differential equations are classified as to the **order** of the highest derivative appearing in an equation. Thus, of the examples cited above, Equations (2)–(5), Art. 67 (*a*), are *equations of the first order;* Equations (2), Ex. 5, Art. 68, and (8), Art. 144, are

equations of the second order; the last equation but one in Ex. 1, Art. 69, is an *equation of the nth order.*

A **solution** (or *integral*) of a differential equation is a relation between the variables which satisfies the equation. Thus, in Art. 73, Ex. 1, relation (1) satisfies Equation (4), and, accordingly, is a solution of (4).

Ex. 1. Show that relation (1) satisfies Equation (4) in Art. 73, Ex. 1.

Ex. 2. See Ex. 4, Art. 79, and Exs. 5–8, Art. 80. In these examples the equations in the ordinary functions are solutions of the differential equations associated with them.

Ex. 3. Show that the relations in Exs. 2–5, Art. 73, are solutions of the differential equations obtained in these respective exercises.

185. Constants of integration. General solution. Particular solutions. It has been seen in Art. 73, Ex. 6, that the elimination of *n arbitrary constants* from a relation between two variables gives rise to a *differential equation of the nth order.* This suggests the inference that the most general solution of a differential equation of the nth order must contain n arbitrary constants. For a proof of this, see *Diff. Eq.*, Art. 3, and Appendix, Note C. Simple instances of this principle have appeared in Art. 73, Exs. 1–5.

A **general solution** of an ordinary differential equation is a solution involving n arbitrary constants. These n constants are called **constants of integration. Particular solutions** are obtained from the general solution by giving the arbitrary constants of integration particular values. The solutions of only a few forms of differential equations, even of equations of the first order, can be obtained.

N.B. For a fuller treatment of the topics in Arts. 184, 185, see *Diff. Eq.*, Chap. I.

EQUATIONS OF THE FIRST ORDER.

186. Equations of the form $f(x)\,dx + F(y)\,dy = 0$. Sometimes equations present themselves in this simple form, or are readily transformable into it; that is, to use the expression commonly used, "the variables are separable." The solution is evidently

$$\int f(x)\,dx + \int F(y)\,dy = c.$$

Ex. 1. Solve $$y\,dx + x\,dy = 0. \tag{1}$$

On separating the variables, $\dfrac{dx}{x} + \dfrac{dy}{y} = 0,$

and integrating, $\log x + \log y = \log c;$

whence $$xy = c. \tag{2}$$

Solution (2) can be obtained directly from (1) on noting that $y\,dx + x\,dy$ is $d\,(xy)$.

Ex. 2. $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0.$ **Ex. 3.** $n(x+a)\,dy + m(y+b)\,dx = 0.$

187. Homogeneous equations. These are equations of the form $P\,dx + Q\,dy = 0$, in which P and Q are homogeneous functions of the same degree in x and y. *The substitution of vx for y* leads to an equation in v and x in which the variables are easily separable.

Ex. 1. $(y^2 - x^2)\,dy + 2\,xy\,dx = 0.$ **Ex. 3.** $y^2\,dx + (xy + x^2)\,dy = 0.$

Ex. 2. $(x^2 + y^2)\,dx + xy\,dy = 0.$ **Ex. 4.** $(y^2 - 2\,xy)\,dx = (x^2 - 2\,xy)\,dy.$

188. Exact differential equations. These are equations of the form
$$P\,dx + Q\,dy = 0, \tag{1}$$
in which the first member is an exact differential (see Art. 109). If P and Q satisfy test (2), Art. 109, then (1) is an exact differential equation, and its solution is
$$\int (P\,dx + Q\,dy) = c.$$

Ex. 1. $x\,dy + y\,dx = 0.$ (See Ex. 1, Art. 186.)

Ex. 2. $(2\,xy + 3)\,dx + (x^2 + 4\,y)\,dy = 0.$

Ex. 3. $(e^x \sin y + 2\,x)\,dx + e^x \cos y\,dy = 0.$

Ex. 4. $(ax - y^2)\,dy = (x^2 - ay)\,dx.$

Integrating factors. Equations that are not exact can be made exact by means of what are termed *integrating factors.* In some cases these factors are easily discoverable.

EXAMPLES.

5. Solve $$x\,dy - y\,dx = 0. \tag{1}$$

The first member does not satisfy the test in Art. 109; thus (1) is not an exact differential equation. Multiplication by $1 \div xy$ gives

$$\frac{dy}{y} - \frac{dx}{x} = 0;$$

whence $\log y - \log x = \log c$, and, accordingly, $y = cx$.

Multiplication by $1 \div x^2$ gives

$$\frac{x\,dy - y\,dx}{x^2} = 0;$$

whence $$\frac{y}{x} = c, \text{ i.e. } y = cx.$$

Similarly, multiplication by $1 \div y^2$ makes (1) integrable.

The multipliers used above are called *integrating factors.* In the following examples these factors can be obtained by inspection.

6. Solve $$(y^2 - x^2)\,dy + 2\,xy\,dx = 0. \quad \text{(See Ex. 1, Art. 187.)}$$

On rearranging, $$y^2\,dy + 2\,xy\,dx - x^2\,dy = 0,$$

and using the factor $1 \div y^2$, $dy + \dfrac{2\,xy\,dx - x^2\,dy}{y^2} = 0.$

Whence, on integration, $$y + \frac{x^2}{y} = c;$$

i.e. $$x^2 + y^2 - cy = 0.$$

7. $2\,ay\,dx = x(y - a)\,dy$.

8. $(y + xy^2)dx = (x^2y - x)dy$.

Note. On *Integrating Factors* see *Diff. Eq.*, Arts. 14–19.

189. The linear equation $$\frac{dy}{dx} + Py = Q, \tag{1}$$

in which P and Q do not involve y. (It is called *linear* because the dependent variable and its derivative appear only in the *first degree.*) This is, perhaps, the most important equation of the first order.

It has been discovered that $e^{\int P\,dx}$ *is an integrating factor* for this equation. On using this factor,

$$e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int P\,dx}; \tag{2}$$

whence, on integration,

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c.$$

Note. For the discovery of the integrating factor, see *Diff. Eq.*, Art. 20.

EXAMPLES.

1. Show that (2) is an exact differential equation.

2. $x\dfrac{dy}{dx} - ay = x + 1.$

On using form (1), $\dfrac{dy}{dx} - \dfrac{a}{x}y = 1 + x^{-1}.$

Here $P = -\dfrac{a}{x}$. $\therefore \int P\,dx = -a\log x = \log x^{-a}$. $\therefore e^{\int P dx} = x^{-a}$.

On using this factor, $x^{-a}(dy - ax^{-1}\,dx) = x^{-a}(1 + x^{-1})\,dx$;

and integrating, $$yx^{-a} = \frac{x^{-a+1}}{1-a} + \frac{x^{-a}}{-a} + c,$$

whence $$y = \frac{x}{1-a} - \frac{1}{a} + cx^a.$$

3. $(1 - x^2)\dfrac{dy}{dx} - xy = 1.$

4. $\cos^2 x\dfrac{dy}{dx} + y = \tan x.$

5. $\dfrac{dy}{dx} + \dfrac{1 - 2x}{x^2}y = 1.$

Some equations are reducible to form (1). For example,

$$\frac{dy}{dx} + Py = Qy^n. \tag{3}$$

On division by y^n, $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q.$

On putting $y^{1-n} = v$, it will be found that (3) takes the linear form

$$\frac{dv}{dx} + (1-n)Pv = (1-n)Q. \tag{4}$$

6. Derive (4) from (3).

7. $\dfrac{dy}{dx} + \dfrac{xy}{1 - x^2} = xy^{\frac{1}{2}}.$

8. $\dfrac{dy}{dx} = x^3y^3 - xy.$

190. Equations not of the first degree in the derivative. Three types of these equations will be considered here, viz. A, B, C, that follow. (Let $\dfrac{dy}{dx}$ be denoted by p.)

A. *Equations reducible to the form* $x = f(y, p)$. (1)

On taking the y-derivatives, $\dfrac{1}{p} = \phi\left(y, p, \dfrac{dp}{dy}\right)$ say. (2)

Possibly, (2) may be solvable and give a relation, say,

$$F(p, y, c) = 0. \tag{3}$$

The p-eliminant between (1) and (3) is the solution. If this eliminant is not easily obtainable, Equations (1) and (3), taken together, may be regarded as the solution, since particular corresponding values of x and y can be obtained by giving p particular values.

Ex. 1. $$x = y + a \log p.$$

On taking the y-derivative, $\frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy}$; whence $1 - p = a\frac{dp}{dy}$.

On integrating, $y = c - a \log (p - 1)$;

and thence $$x = c + a \log \frac{p-1}{p}.$$

Ex. 2. $p^2 y + 2px = y$. Ex. 3. $x = y + p^2$.

B. ***Equations reducible to the form*** $\boldsymbol{y = f(x, p)}$. (4)

On taking the x-derivative, $p = \phi\left(x, p, \frac{dp}{dx}\right)$ say. (5)

Possibly, (5) may be solvable and give a relation, say,

$$F(p, x, c) = 0. \tag{6}$$

The p-eliminant between (4) and (6) is the required solution. If this eliminant is not easily obtainable, Equations (4) and (6), taken together, may be regarded as the solution, since they suffice for the determination of x and y by assigning values to a parameter p.

Ex. 4. $4y = x^2 + p^2$. Ex. 5. $2y + p^2 = 2x^2$.

C. Clairaut's equation, viz. $\boldsymbol{y = px + f(p)}$. (7)

In this case $$y = cx + f(c) \tag{8}$$
is obviously a solution.

This solution can be obtained on treating (7) like (4), of which it is a special case.

Thus, on taking the x-derivatives in (7),

$$p = p + [x + f'(p)]\frac{dp}{dx}.$$

From this, $x + f'(p) = 0$ (9), or $\frac{dp}{dx} = 0$. (10)

Equation (10) gives $p = c$.

Substitution of this in (7) gives (8).

As to the part played by (9) see *Diff. Eq.*, Art. 34.

EXAMPLES.

6. $y = px + \dfrac{a}{p}$. 7. $y = px + a\sqrt{1+p^2}$.

8. $x^2(y - px) = yp^2$. [Suggestion: Put $x^2 = u$, $y^2 = v$.]

NOTE 1. Sometimes the first member of an equation $f(x, y, p) = 0$ is *resolvable into factors*. In such a case equate each factor to zero, and solve the equation thus made. (This is analogous to the method pursued in solving rational algebraic equations involving one unknown.)

9. Solve $p^3 - p^2(x + y + 2) + p(xy + 2x + 2y) - 2xy = 0$.

On factoring, $(p - x) = 0,\ p - y = 0,\ p - 2 = 0.$

On solving, $2y = x^2 + c,\ y = ce^x,\ y = 2x + c.$

These solutions may be combined together,

$$(2y - x^2 - c)(y - ce^x)(y - 2x - c) = 0.$$

NOTE 2. On *Equations of the first order which are not of the first degree* see *Diff. Eq.*, Chap. III.

191. Singular solutions. Let a differential equation $f(x, y, p) = 0$ have a solution $f(x, y, c) = 0$. The latter is geometrically represented by a family of curves. The equation of the envelope of this family (Art. 154) is termed *the singular solution* of the differential equation. That the equation of the envelope is a solution is evident from the definition of an envelope (see Art. 154) and this fact, viz. that at any point on any one of the curves of the family the coördinates of the point and the slope of the curve satisfy the differential equation. The singular solution is obviously distinct from the general solution and from any particular solution.

For example, the general solution [(8), Art. 190] of Clairaut's equation is, geometrically, a family of straight lines. The envelope of this family of lines is the singular solution of (7). The envelope of (8) may be obtained by the method shown in Art. 157. Differentiation of the members of (8) with respect to c gives

$$0 = x + f'(c).$$

The envelope is the c-eliminant between this equation and (8).

EXAMPLES.

1. Show that the singular solution of Ex. 6, Art. 190, is $y^2 = 4ax$.

2. Find the singular solutions of the equations in Exs. 7, 8, Art. 190.

3. Find the general solution and the singular solution of:

(1) $y = px + p^2$. (2) $p^2x = y$. (3) $8a(1+p)^3 = 27(x+y)(1-p)^3$.

NOTE 1. The singular solution can also be derived directly from the differential equation, without finding the general solution; see reference below.

NOTE 2. On *Singular Solutions* see *Diff. Eq.*, Chap. IV., pages 40–49.

192. Orthogonal Trajectories. Associated with a family of curves (Art. 154), there may be another family whose members intersect the members of the first family at right angles. An instance is given in Ex. 1. The members of the one family are said to be *orthogonal trajectories* of the other family.

For example, the orthogonal trajectories of a family of concentric circles are the straight lines passing through the common centre of the circles.

***A*. To find the orthogonal trajectories of the family**

$$f(x, y, a) = 0, \tag{1}$$

in which a is the arbitrary parameter. Let the differential equation of this family, which is obtained by the elimination of a (see Art. 73), be

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0. \tag{2}$$

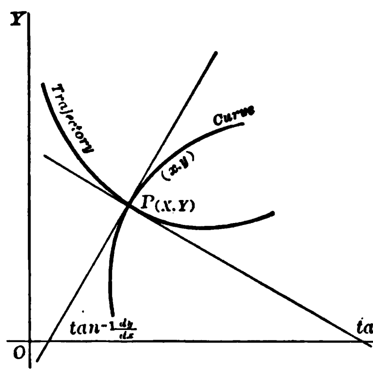

FIG. 102.

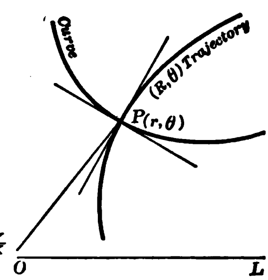

FIG. 103.

Let P be any point, through which pass a curve of the family and an orthogonal trajectory of the family, as shown in Fig. 102. For the moment, for the sake of distinction, let (x, y) denote the coördinates of P regarded as a point on the given curve, and let

(X, Y) denote the coördinates of P regarded as a point on the trajectory. At P the slope of the tangent to the curve and the slope of the tangent to the trajectory are respectively $\frac{dy}{dx}$ and $\frac{dY}{dX}$. Since these tangents are at right angles to each other,

$$\frac{dy}{dx} = -\frac{dX}{dY}.$$

Also $$x = X, \text{ and } y = Y.$$

Substitution in (2) gives

$$\phi\left(X, Y, -\frac{dX}{dY}\right) = 0. \tag{3}$$

But $P(X, Y)$ is any point on any trajectory. Accordingly, (3) or, what is the same equation,

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0 \tag{3'}$$

is the differential equation of the orthogonal trajectories of the curves (1) or (2).

Hence: *To find the differential equation of the family of orthogonal trajectories of a given family of curves substitute* $-\frac{dx}{dy}$ *for* $\frac{dy}{dx}$ *in the differential equation of the given family.*

EXAMPLES.

1. Find the orthogonal trajectories of the family of circles which pass through the origin and have their centres on the x-axis.

The equation of these circles is

$$x^2 + y^2 = 2ax, \tag{4}$$

in which a is the arbitrary parameter.

On differentiation and the elimination of a (Art. 73), there is obtained the differential equation of the family, viz.

$$y^2 - x^2 - 2xy\frac{dy}{dx} = 0. \tag{5}$$

The substitution of $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ gives the differential equation of the orthogonal curves, viz.

$$y^2 - x^2 + 2xy\frac{dx}{dy} = 0. \tag{6}$$

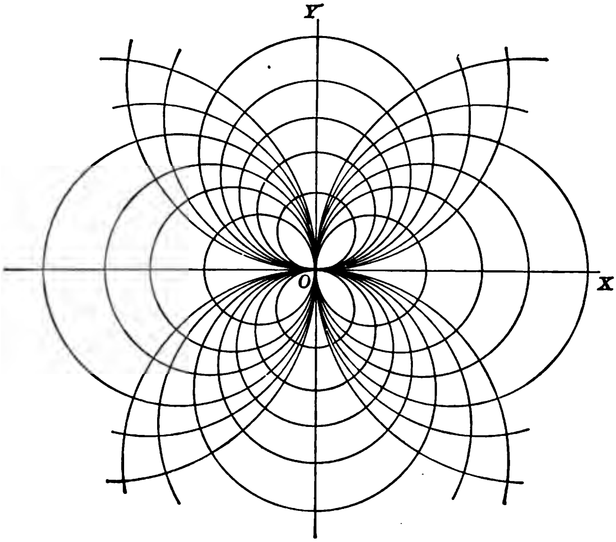

FIG. 104.

Integration of (6) [see Art. 188, Ex. 6] gives

$$x^2 + y^2 = cy, \tag{7}$$

the orthogonal family, viz. a family of circles passing through the origin and having their centres on the y-axis. (See Fig. 104.)

2. Obtain the orthogonal trajectories of the circles (7), viz. the circles (4).

3. *Derive* the equation of the orthogonal trajectories of the family of lines $y = mx$.

4. *Derive* the equation of the family of concentric circles whose centre is at the origin.

***B.* To find the orthogonal trajectories of the family**

$$f(r, \theta, c) = 0, \tag{8}$$

in which c is the arbitrary parameter. Let the differential equation of this family, which is obtained by the elimination of c, be

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0. \tag{9}$$

Let P be any point through which pass a curve of the given family and an orthogonal trajectory of the family, as shown in Fig. 103. For the moment, for the sake of distinction, let (r, θ) denote the coördinates of P regarded as a point on the given curve, and let (R, Θ) denote the coördinates of P regarded as a point on the trajectory. At P (see Art. 60) the tangent to the given curve and the tangent to the trajectory make with the radius vector angles whose tangents are respectively $r\dfrac{d\theta}{dr}$ and $R\dfrac{d\Theta}{dR}$.

Since these tangent lines are at right angles to each other,

$$r\frac{d\theta}{dr} = -\frac{1}{R\dfrac{d\Theta}{dR}}; \text{ whence } \frac{dr}{d\theta} = -rR\frac{d\Theta}{dR} = -R^2\frac{d\Theta}{dR}.$$

Accordingly (9) may be written

$$F\left(R, \Theta, -R^2\frac{d\Theta}{dR}\right) = 0. \tag{10}$$

But $P(R, \Theta)$ is any point on any trajectory. Accordingly (10), or the same expression in the usual symbols r and θ,

$$F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0, \tag{10$'$}$$

is the differential equation of the orthogonal trajectories of the curves (8) or (9).

Hence: *To find the differential equation of the family of orthogonal trajectories of a given family of curves, substitute* $-r^2\dfrac{d\theta}{dr}$ *for* $\dfrac{dr}{d\theta}$ *in the differential equation of the given family.*

EXAMPLES.

5. Find the orthogonal trajectories of the set of circles $r = a\cos\theta$, a being the parameter.

Differentiation and the elimination of a gives the differential equation of these circles, viz.

$$\frac{dr}{d\theta} + r\tan\theta = 0.$$

On substituting, as directed above, there is obtained

$$r\frac{d\theta}{dr} = \tan\theta,$$

the differential equation of the orthogonal trajectories. Integration gives another family of circles

$$r = c\sin\theta, \tag{11}$$

6. Sketch the families of circles in Ex. 5, and show that the problem and result in Ex. 5 are practically the same as the problem and result in Ex. 1.

7. Find the orthogonal trajectories of circles (11), viz. the circles in Ex. 5.

N.B. Various geometrical problems requiring differential equations are given in the following examples.

NOTE 1. On *applications of differential equations of the first order*, see *Diff. Eq.*, Chap. V.

8. Find the curves respectively orthogonal to each of the following families of curves (*sketch the curves and their trajectories*): (1) the parabolas $y^2 = 4\,ax$; (2) the hyperbolas $xy = k^2$; (3) the curves $a^{n-1}y = x^n$; interpret the cases $n = 0, 1, -1, 2, -2, \pm\frac{1}{2}, \pm\frac{3}{2}$, respectively; (4) the hypocycloids $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; (5) the parabolas $y = ax^2$; (6) the cardioids $r = a(1 - \cos\theta)$; (7) the curves $r^n \sin n\theta = a^n$; (8) the curves $r^n = a^n \cos n\theta$; (9) the lemniscates $r^2 = a^2 \cos 2\,\theta$; (10) the confocal and coaxial parabolas $r = \dfrac{2\,a}{1 + \cos\theta}$; (11) the circles $x^2 + y^2 + 2\,my = a^2$, in which m is the parameter.

9. (*a*) Show that the differential equation of the confocal parabolas $y^2 = 4\,a(x + a)$ is the same as the differential equation of the orthogonal curves, and interpret the result. (*b*) Show that the differential equation of the confocal conics $\dfrac{x^2}{a^2 + l} + \dfrac{y^2}{b^2 + l} = 1$ is the same as the differential equation of the orthogonal curves, and interpret the result.

10. Find the curve such that the product of the lengths of the perpendiculars drawn from two fixed points to any tangent is constant.

11. Find the curve such that the product of the lengths of the perpendiculars drawn from two fixed points to any normal is constant.

12. Find the curve such that the tangent intercepts on the perpendiculars to the axis of x at the points $(a, 0)$, $(-a, 0)$, lengths whose product is b^2.

13. Find the curve such that the product of the lengths of the intercepts made by any tangent on the coördinate axes, is equal to a constant a^2.

14. Find the curve such that the sum of the intercepts made by any tangent on the coördinate axes is equal to a constant a.

EQUATIONS OF THE SECOND AND HIGHER ORDERS.

Only a very few classes of these equations will be solved here; namely, simple forms of linear equations with constant coefficients and homogeneous linear equations. Three special equations of the second order will also be briefly discussed.

193. Linear Equations. Linear equations are those which are of the first degree in the dependent variable and its derivatives. The general type of these equations is

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = X,$$

in which $P_1, P_2, \cdots, P_n, X$, do not involve y or its derivatives.

(For some general properties of these equations see Murray, *Integral Calculus*, Art. 113, *Diff. Eq.*, Art. 49.)

***A*. The linear equation** $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = 0$, (1)

in which the coefficients $P_1, P_2, \cdots, P_n$, are constants.

The substitution of e^{mx} for y in the first member, gives

$$(m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n)e^{mx}.$$

This expression is zero for all values of m that satisfy the equation

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0; \qquad (2)$$

and, accordingly, for each of these values of m, $y = e^{mx}$ is a solution of (1). Equation (2) is called the *auxiliary equation.* Let $m_1, m_2, \cdots, m_n$, be its roots. Substitution will show that $y = c_1e^{m_1x}$, $y = c_2e^{m_2x}, \cdots, y = c_ne^{m_nx}$, and also

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}, \qquad (3)$$

in which the c's are arbitrary constants, are solutions of (1). Solution (3) contains n arbitrary constants and, accordingly, is the general solution.

NOTE 1. *If two roots of* (2) *are imaginary*, say $\alpha + i\beta$ and $\alpha - i\beta$, i denoting $\sqrt{-1}$, the corresponding solution is

$$y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}.$$

According to Art. 179 this may be put in the form

$$\begin{aligned} y &= e^{\alpha x}(c_1e^{i\beta x} + c_2e^{-i\beta x}) \\ &= e^{\alpha x}\{c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\}, \\ &= e^{\alpha x}\{(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x\}, \\ &= e^{\alpha x}(A\cos\beta x + B\sin\beta x), \end{aligned}$$

in which A and B are arbitrary constants, since c_1 and c_2 are arbitrary constants.

NOTE 2. *If two roots of* (2) *are equal*, say m_1 and m_2 each equal to a, the corresponding solution, viz.

$$y_1 = c_1e^{m_1x} + c_2e^{m_2x},$$

becomes $$y = (c_1 + c_2)e^{ax}, \text{ i.e. } y = ce^{ax},$$

which does not involve two arbitrary constants. Put $m_2 = a + h$; then the solution takes the form

$$y = c_1e^{ax} + c_2e^{(a+h)x},$$
$$= e^{ax}(c_1 + c_2e^{hx}).$$

On expanding e^{hx} in the exponential series (Art. 178, Ex. 7), this equation becomes

$$y = e^{ax}(A + Bx + \tfrac{1}{2}c_2h^2x^2 + \text{terms in ascending powers of } h), \qquad (4)$$

in which $A = c_1 + c_2$ and $B = c_2h$. On letting h approach zero in (4), the latter becomes

$$y = e^{ax}(A + Bx).$$

(The numbers c_1 and c_2 can always be chosen so that $c_1 + c_2$ and c_2h are finite.)

If a root a of (2) is repeated r times, the corresponding solution is

$$y = (c_1 + c_2x + c_3x^2 + \cdots + c_rx^{r-1})e^{ax}.$$

NOTE 3. On Equation (1), see *Diff. Eq.*, Arts. 50–55.

EXAMPLES.

1. Solve $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = 0.$

The auxiliary equation is $m^3 - 3m + 2 = 0$;

its roots are $-2, 1, 1.$

Accordingly, the solution is $y = c_1e^{-2x} + (c_2 + c_3x)e^{x}.$

2. Solve $\dfrac{d^2y}{dx^2} + a^2y = 0.$

The auxiliary equation is $m^2 + a^2 = 0$;

its roots are $ai, -ai.$

Accordingly, its solution is $y = c_1e^{aix} + c_2e^{-aix}$

$= A\cos ax + B\sin ax.$ (See Ex. 1, Art. 73.)

3. Solve the following differential equations:

(1) $D^2y - 4Dy + 13y = 0.$ (2) $D^3y - 7Dy + 6y = 0.$

(3) $\dfrac{d^3y}{dx^3} - 12\dfrac{dy}{dx} - 16y = 0.$ (4) $\dfrac{d^4y}{dx^4} - 10\dfrac{d^3y}{dx^3} + 62\dfrac{d^2y}{dx^2} - 160\dfrac{dy}{dx} + 136y = 0.$

B. The "homogeneous" linear equation

$$x^n\frac{d^ny}{dx^n}+p_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+p_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\cdots+p_ny=0, \qquad (5)$$

in which $p_1, p_2, \cdots, p_n$, *are constants.*

First method of solution. If the independent variable x be changed to z by means of the relation

$$z=\log x, \text{ i.e. } x=e^z,$$

the equation will be transformed into an equation with constant coefficients. (For examples, see Art. 92 and Exs. 3 (i), (v), (vi), page 147.)

4. Show the truth of the statement last made.

5. Solve Exs. 7 below by this method.

Second method of solution. The substitution of x^m for y in the first member of equation (5) gives

$$[m(m-1)\cdots(m-n+1)+p_1m(m-1)\cdots(m-n+2)+\cdots+p_n]x^m.$$

This is zero for all values of m that satisfy the equation

$$m(m-1)\cdots(m-n+1)+p_1m(m-1)\cdots(m-n+2)+\cdots+p_n=0. \qquad (6)$$

Let the roots of (6) be $m_1, m_2, \cdots, m_n$; then it can be shown, as in the case of solution (3) and equation (1), that

$$y=c_1x^{m_1}+c_2x^{m_2}+\cdots+c_nx^{m_n}$$

is the general solution of equation (5).

The forms of this solution, when the auxiliary equation (6) has repeated roots or imaginary roots, will become apparent on solving equation (5) by the first method.

EXAMPLES.

6. Show that the solution of (5) corresponding to an r-tuple root m of (6), is $y=x^m[c_1+c_2\log x+c_3(\log x)^2+\cdots+c_r(\log x)^{r-1}]$; and show that the solution of (5) corresponding to two imaginary roots $\alpha+i\beta$, $\alpha-i\beta$, of (6), is

$$y=x^\alpha[c_1\cos(\beta\log x)+c_2\sin(\beta\log x)].$$

7. Solve the following equations:

(1) $x^2D^2y - xDy + 2y = 0.$ (2) $x^2D^2y - xDy + y = 0.$

(3) $x^2D^2y - 3xDy + 4y = 0.$ (4) $x^3D^3y + 2x^2D^2y + 2y = 0.$

NOTE 3. Equations of the form

$$(a + bx)^n\frac{d^ny}{dx^n} + p_1(a + bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + p_2(a + bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_ny = 0$$

are reducible to the homogeneous linear form, by putting $a + bx = z$.

8. Show the truth of the last statement.

9. Solve $(5 + 2x)^2\frac{d^2y}{dx^2} - 6(5 + 2x)\frac{dy}{dx} + 8y = 0.$

NOTE 4. On Equation (5), see *Diff. Eq.*, Arts. 65, 66, 71.

194. Special equations of the second order.

A. Equations of the form $\dfrac{d^2y}{dx^2} = f(y)$.

For these equations $2\dfrac{dy}{dx}$ is an integrating factor.

EXAMPLES.

1. $\dfrac{d^2y}{dx^2} + a^2y = 0.$ (See Ex. 2, Art. 193.)

On using the factor $2\dfrac{dy}{dx}$, $2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = -2a^2y\dfrac{dy}{dx}.$

On integrating, $\left(\dfrac{dy}{dx}\right)^2 = -a^2y^2 + k$

$= a^2(c^2 - y^2)$, on putting a^2c^2 for k.

On separating the variables, $\dfrac{dy}{\sqrt{c^2-y^2}} = a\,dx,$

and integrating, $\sin^{-1}\dfrac{y}{c} = ax + \alpha.$

This result may be written $y = c\sin(ax + \alpha),$

or $y = A\sin ax + B\cos ax.$

2. Show the equivalence of the last two forms. Express A and B in terms of c and α, and express c and α in terms of A and B.

3. Show that $2\dfrac{dy}{dx}$ is an integrating factor in case *A*.

4. Solve the following equations:

(1) $\dfrac{d^2y}{dx^2} = a^2y.$ (2) $\dfrac{d^2y}{dx^2} = e^{2y}.$

(3) If $\dfrac{d^2s}{dt^2} = \dfrac{-k}{s^2}$, find t, given that $\dfrac{dx}{dt} = 0$ and $x = a$, when $t = 0$.

B. Equations of the form $f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0.$ (1)

On letting p denote $\frac{dy}{dx}$, this may be written $f\left(\frac{dp}{dx}, p, x\right) = 0.$ (2)

Integration of (2) may give $\phi(p, x, c) = 0$,

and this may happen to be integrable.

EXAMPLES.

5. Find the curve whose radius of curvature is constant and equal to a. (This example is the converse of Art. 147.)

6. Solve the following equations:

(1) $(1 + x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0.$ (3) $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$

(2) $xD^2y + Dy = 0.$ (4) $(1 + x)D^2y + Dy + x = 0.$

C. Equations of the form $f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0.$ (1)

This (see Art. 90) may be written

$$f\left(p\frac{dp}{dy}, p, y\right) = 0. \tag{2}$$

Integration of (2) may give

$$F(p, y, c) = 0,$$

and this may happen to be integrable.

EXAMPLES.

7. Solve $$\frac{d^2y}{dx^2} + a^2y = 0. \quad \text{(See Ex. 1.)}$$

This is $$p\frac{dp}{dy} = -a^2y.$$

Now proceed as in Ex. 1.

8. Solve the following equations:

(1) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1.$ (2) $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2 \log y.$

(3) $y^3D^2y + 1 = 0.$ (4) $D^2y + (Dy)^2 + 1 = 0.$

NOTE 5. For the solution of equations in the form $D^ny = f(x)$, see Art. 129.

NOTE 6. On forms like A, B, C, see *Diff. Eq.*, Arts. 77, 78, 79, respectively.

NOTE 7. **References for collateral reading.** For a brief treatment of differential equations and for interesting practical examples, see Lamb, *Calculus*, Chaps. XI., XII. (pp. 456–540); also see F. G. Taylor, *Calculus*, Chaps. XXIX.–XXXIV. (pp. 493–564), and Gibson, *Calculus*, Chap. XX. (pp. 424–441).

EXAMPLES.

Solve the following equations:

(1) $r d\theta = \tan\theta \, dr$. (2) $(1+y)dx + x(x+y)dy = 0$.

(3) $(4y+3x)dy+(y-2x)dx=0$. (4) $x\frac{dy}{dx}-y=\sqrt{x^2+y^2}$. (5) $\frac{dy}{dx}+y\tan x=1$.

(6) $x\frac{dy}{dx}-2y=x^4\sqrt{1+x^2}$. (7) $(6x+4y+5)dx+(10y+4x+1)dy=0$.

(8) $y(y\,dx-x\,dy)+x\sqrt{x^2+y^2}\,dy=0$. (9) $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{(x^2+1)^3}$.

(10) $3\frac{dy}{dx}+\frac{2}{x+1}y=\frac{x^3}{y^2}$. (11) $x-yp=ap^2$. (12) $y^2=a^2(1+p^2)$.

(13) $(px-y)(py+x)=h^2p$. (14) $p^2x^3+x^2py=1$. (15) $x=2y-3p^2$.

(16) $p^2+2py\cot x=y^2$. (17) $y\sqrt{1+p^2}=a$; also find the singular solution.

(18) $y-px=\sqrt{b^2+a^2p^2}$; also find the singular solution. (19) $xp^2=(x-a)^2$, and also find the singular solution. (20) $\frac{d^4y}{dx^4}-a^4y=0$. (21) $\frac{d^4y}{dx^4}+4y=0$.

(22) $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4y=0$. (23) $x^2\frac{d^2y}{d^2x}+x\frac{dy}{dx}-y=0$.

(24) $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{yd}{dx}+y=0$. (25) $(x+a)^2\frac{d^2y}{d^2x}-4(x+a)\frac{dy}{dx}+6y=0$.

(26) $y^3\frac{d^2y}{dx^2}=a$. (27) $\left(\frac{d^2y}{dx^2}\right)^2=\frac{1}{ay}$. (28) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=2x$. (29) $\left(\frac{d^2y}{dx^2}\right)=a\left(\frac{dy}{dx}\right)^2$.

APPENDIX.

NOTE A.

ON HYPERBOLIC FUNCTIONS.

1. This note gives a short account of hyperbolic functions and their properties. The student will probably meet these functions in his reading; for many results in pure and applied mathematics can be expressed in terms of them, and their values are tabulated for certain ranges of numbers.* There are close analogies between the hyperbolic functions and the circular (or trigonometric) functions (*a*) in their algebraic definitions, (*b*) in their connection with certain integrals, (*c*) in their respective relations to the rectangular hyperbola and the circle.

2. Names, symbols, and algebraic definitions of the hyperbolic functions. The hyperbolic functions of a number x are its hyperbolic sine, hyperbolic cosine, hyperbolic tangent, ⋯, hyperbolic cosecant, and the corresponding six inverse functions. These functions have been respectively denoted by the symbols $\sinh x$, $\cosh x$, $\tanh x$, $\coth x$, $\operatorname{sech} x$, $\operatorname{cosech} x$, $\sinh^{-1} x$, etc. These are the symbols in common use. As to symbols for the hyperbolic functions, the following suggestion has been made by Professor George M. Minchin in *Nature*, Vol. 65 (April 10, 1902), page 531: "If the prefix *hy* were put to each of the trigonometrical functions, all the names would be pronounceable and not too long. Thus, *hysin* x, *hytan* x, etc., would at once be pronounceable and indicate the

* See tables of the hyperbolic functions of numbers in Peirce, *Short Table of Integrals* (revised edition, 1902), pages 120–123; Lamb, *Calculus*, Table E, page 611; Merriman and Woodward, *Higher Mathematics*, pages 162–168.

hyperbolic nature of the functions." This notation will be adopted in this note.*

The direct hyperbolic functions are **algebraically** defined as follows:

$$\textbf{hysin}\, x = \frac{e^x - e^{-x}}{2}, \qquad \textbf{hycos}\, x = \frac{e^x + e^{-x}}{2},$$

$$\textbf{hytan}\, x = \frac{\textbf{hysin}\, x}{\textbf{hycos}\, x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \textbf{hycot}\, x = \frac{\textbf{hycos}\, x}{\textbf{hysin}\, x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \qquad (1)$$

$$\textbf{hysec}\, x = \frac{1}{\textbf{hycos}\, x}, \qquad \textbf{hycosec}\, x = \frac{1}{\textbf{hysin}\, x}.$$

There is evidently a close analogy between these definitions and the definitions and properties of the circular functions. [See the exponential expressions (or definitions) for $\sin x$ and $\cos x$ in Art. 179.]

From the definitions for hysin x and hycos x can be deduced, by means of the expansions for e^x and e^{-x} (see Art. 178, Ex. 7), the following *series*, which are analogous to the series for $\sin x$ and $\cos x$ (Art. 178, Exs. 2, 5):

$$\textbf{hysin}\, x = x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \cdots,$$

$$\textbf{hycos}\, x = 1 + \frac{x^2}{2\,!} + \frac{x^4}{4\,!} + \cdots; \qquad (2)$$

The second members in equations (2) may be regarded as definitions of hysin x and hycos x.

EXAMPLES.

1. Derive the following relations, both from the exponential definitions of $\sin x$, $\cos x$, hysin x, hycos x, and from the expansions of these functions in series: (1) $\cos x =$ hycos (ix); (2) $i \sin x =$ hysin (ix); (3) $\cos (ix) =$ hycos x; (4) $\sin (ix) = i$ hysin x.

2. (*a*) Show that $e^x =$ hycos $x +$ hysin x, $e^{-x} =$ hycos $x -$ hysin x. [Compare Art. 179 (1), (2).] (*b*) Show that hysin $0 = 0$, hycos $0 = 1$, hytan $0 = 0$, hysin $\infty = \infty$, hycos $\infty = \infty$, hytan $\infty = 1$, hysin $(-x) = -$ hysin x, hycos $(-x) =$ hycos x, hytan $(-x) = -$ hytan x.

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* The symbols used in W. B. Smith's *Infinitesimal Analysis* are *hs*, *hc*, *ht*, *hct*, *hsc*, *hcsc*.

3. Show that the following relations exist between the hyperbolic functions:

(1) $\text{hycos}^2 x - \text{hysin}^2 x = 1$; (2) $\text{hysec}^2 x + \text{hytan}^2 x = 1$;
(3) $\text{hysin}\,(x \pm y) = \text{hysin}\,x \cdot \text{hycos}\,y \pm \text{hycos}\,x \cdot \text{hysin}\,y$;
(4) $\text{hycos}\,(x \pm y) = \text{hycos}\,x \cdot \text{hycos}\,y \pm \text{hysin}\,x \cdot \text{hysin}\,y$;
(5) $\text{hytan}\,(x \pm y) = (\text{hytan}\,x \pm \text{hytan}\,y) \div (1 \pm \text{hytan}\,x \cdot \text{hytan}\,y)$;
(6) $\text{hysin}\,2x = 2\,\text{hysin}\,x \cdot \text{hycos}\,x$;
(7) $\text{hycos}\,2x = \text{hycos}^2 x + \text{hysin}^2 x = 2\,\text{hycos}^2 x - 1 = 1 + 2\,\text{hysin}^2 x$;
(8) $\text{hytan}\,2x = 2\,\text{hytan}\,x \div (1 + \text{hytan}^2 x)$.

Compare these relations with the corresponding relations between the circular functions.

4. Show the following: (1) $\dfrac{d(\text{hysin}\,x)}{dx} = \text{hycos}\,x$; (2) $\dfrac{d(\text{hycos}\,x)}{dx} = \text{hysin}\,x$; (3) $\dfrac{d(\text{hytan}\,x)}{dx} = \text{hysec}^2 x$; (4) $\dfrac{d(\text{hycot}\,x)}{dx} = -\text{hycsc}^2 x$; (5) $\dfrac{d(\text{hysec}\,x)}{dx} = -\,\text{hysec}\,x \cdot \text{hytan}\,x$; (6) $\dfrac{d(\text{hycsc}\,x)}{dx} = -\,\text{hycsc}\,x \cdot \text{hycot}\,x$; (7) $\int \text{hysin}\,x\,dx = \text{hycos}\,x$; (8) $\int \text{hycos}\,x\,dx = \text{hysin}\,x$; (9) $\int \text{hytan}\,x\,dx = \log\,(\text{hycos}\,x)$; (10) $\int \text{hycot}\,x\,dx = \log\,(\text{hysin}\,x)$; (11) $\int \text{hysec}\,x\,dx = 2\tan^{-1} e^x$; (12) $\int \text{hycsc}\,x\,dx = \log\left(\text{hytan}\,\dfrac{x}{2}\right)$. Compare these relations with the corresponding relations between the circular functions.

5. Make graphs of the functions $\text{hysin}\,x$, $\text{hycos}\,x$, $\text{hytan}\,x$. (See Lamb, *Calculus*, pp. 42, 43.)

6. Show that the slope of the catenary $\dfrac{y}{a} = \text{hycos}\,\dfrac{x}{a}$ is $\text{hysin}\,\dfrac{x}{a}$. Sketch this curve.

Inverse hyperbolic functions. The statement "the hyperbolic sine of y is x" is equivalent to the statement "y is a number whose hyperbolic sine is x." These statements are expressed in mathematical shorthand,

$$\textbf{hysin}\,y = x, \quad y = \textbf{hysin}^{-1}\,x. \tag{3}$$

The last symbol is read "the inverse hyperbolic sine of x," or "the anti-hyperbolic sine of x." The other inverse hyperbolic functions are defined and symbolised in a similar manner.

The inverse hyperbolic functions can also be expressed in terms of logarithmic functions, and thus they may be given **logarithmic definitions.** (This might have been expected, for the direct hyperbolic functions are defined in terms of exponential functions, and the logarithm is the inverse of the exponential.)

Let $\qquad \text{hysin}\, y = x$; then $x = \frac{1}{2}(e^y - e^{-y})$.

This equation reduces to $e^{2y} - 2xe^y - 1 = 0$.

On solving for e^y, $\qquad e^y = x + \sqrt{x^2+1}$. $\qquad$ (4)

(For real values of y, e^y being positive, the *positive* sign of the radical must be taken.)

From (4) $\qquad \mathbf{y = hysin^{-1}\, x = log(x + \sqrt{x^2+1})}$. $\qquad$ (5)

N.B. The base of the logarithms in this note is e.

In a similar manner, on putting

$$x = \text{hycos}\, y = \tfrac{1}{2}(e^y + e^{-y}),$$

and solving for e^y,

$$e^y = x \pm \sqrt{x^2-1}. \tag{6}$$

For real values of y, x is greater than 1 and both signs of the radical can be taken.

From (6) and the fact that $(x + \sqrt{x^2-1})(x - \sqrt{x^2-1}) = 1$, and thus $\log(x - \sqrt{x^2-1}) = -\log(x + \sqrt{x^2-1})$, it follows that

$$\mathbf{y = hycos^{-1}\, x = \pm\, log(x + \sqrt{x^2-1})}. \tag{7}$$

In a similar manner it can be shown that

$$\mathbf{hytan^{-1}\, x = \frac{1}{2} \log \frac{1+x}{1-x}}, \tag{8}$$

where $x^2 < 1$ for real values of $\text{hytan}^{-1} x$; and that

$$\mathbf{hycot^{-1}\, x = \frac{1}{2} \log \frac{x+1}{x-1}}, \tag{9}$$

where $x^2 > 1$ for real values of $\text{hycot}^{-1} x$.

EXAMPLES.

7. Derive the relations (7), (8), (9).

8. Solve equations (5), (7), (8), (9), for x in terms of y, and thus obtain the definitions of the direct hyperbolic functions.

9. Show that the differentials of $\text{hysin}^{-1} x$, $\text{hycos}^{-1} x$, $\text{hytan}^{-1} x$, $\text{hycot}^{-1} x$, are respectively $\dfrac{dx}{\sqrt{x^2+1}}$, $\pm \dfrac{dx}{\sqrt{x^2-1}}$, $\dfrac{dx}{1-x^2}$ for $x^2 < 1$, $-\dfrac{dx}{x^2-1}$ for $x^2 > 1$. Compare these with the differentials of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$.

10. Following the method by which relations (5)–(9) have been derived, show that:

$$\text{hysin}^{-1}\frac{x}{a}=\log\frac{x+\sqrt{x^2+a^2}}{a}; \qquad \text{hycos}^{-1}\frac{x}{a}=\pm\log\frac{x+\sqrt{x^2-a^2}}{a};$$

$$\text{hytan}^{-1}\frac{x}{a}=\frac{1}{2}\log\frac{a+x}{a-x}\text{ for }x^2<a^2; \quad \text{hycot}^{-1}\frac{x}{a}=\frac{1}{2}\log\frac{x+a}{x-a}\text{ for }x^2>a^2.$$

11. Assuming the relations in Ex. 10, show that the x-differentials are:

$$d\left(\text{hysin}^{-1}\frac{x}{a}\right)=\frac{dx}{\sqrt{x^2+a^2}}; \qquad d\left(\text{hycos}^{-1}\frac{x}{a}\right)=\pm\frac{dx}{\sqrt{x^2-a^2}};$$

$$d\left(\text{hytan}^{-1}\frac{x}{a}\right)=\frac{a\,dx}{a^2-x^2}\text{ for }x^2<a^2; \quad d\left(\text{hycot}^{-1}\frac{x}{a}\right)=-\frac{a\,dx}{x^2-a^2}\text{ for }x^2>a^2.$$

Compare these differentials with the differentials of $\sin^{-1}\frac{x}{a}$, $\cos^{-1}\frac{x}{a}$, $\tan^{-1}\frac{x}{a}$, $\cot^{-1}\frac{x}{a}$.

12. Assuming the relations in Ex. 10 as definitions of the inverse hyperbolic functions, derive the definitions of the corresponding direct hyperbolic functions. (Suggestion. Follow the plan outlined in Ex. 8.)

3. Inverse hyperbolic functions defined as integrals. It follows from Ex. 11. Art. 2, that

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\text{hysin}^{-1}\frac{x}{a}+c; \quad \int\frac{dx}{\sqrt{x^2-a^2}}=\pm\text{ hycos}^{-1}\frac{x}{a}+c;$$

$$\int\frac{dx}{a^2-x^2}=\frac{1}{a}\text{hytan}^{-1}\frac{x}{a}+c,\ (x^2<a^2); \quad \int\frac{dx}{x^2-a^2}=-\frac{1}{a}\text{hycot}^{-1}\frac{x}{a}+c,$$

$(x^2>a^2)$.

Accordingly, these inverse hyperbolic functions can be expressed in terms of certain definite integrals, viz.:

$$\int_0^u\frac{dx}{\sqrt{x^2+a^2}}=\log\frac{u+\sqrt{u^2+a^2}}{a}=\text{hysin}^{-1}\frac{u}{a};$$

$$\int_a^u\frac{dx}{\sqrt{x^2-a^2}}=\log\frac{u+\sqrt{u^2-a^2}}{a}=\pm\text{ hycos}^{-1}\frac{u}{a};$$

$$\int_0^u\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+u}{a-u}=\frac{1}{a}\text{hytan}^{-1}\frac{u}{a},\ u^2<a^2;$$

$$\int_\infty^u\frac{dx}{x^2-a^2}=-\frac{1}{2a}\log\frac{u+a}{u-a}=-\frac{1}{a}\text{hycot}^{-1}\frac{u}{a},\ u^2>a^2.$$

These relations between definite integrals and inverse hyperbolic functions may be taken as *definitions* of the functions.

The inverse circular functions can be defined by integrals which are very similar to the integrals appearing in the definitions of the hyperbolic functions. Thus:

$$\int_0^u \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{u}{a}, \qquad \int_a^u \frac{dx}{\sqrt{a^2 - x^2}} = -\cos^{-1}\frac{u}{a},$$

$$\int_0^u \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}, \qquad \int_\infty^u \frac{dx}{a^2 + x^2} = -\frac{1}{a}\cot^{-1}\frac{u}{a}.$$

EXAMPLES.

1. Find the area of the sector AOP of the hyperbola $x^2 - y^2 = a^2$ (Fig. 106), P being the point for which $x = u$. Thence show, from the definition above, that $\text{hycos}^{-1}\frac{u}{a}$ is **the ratio** *of twice the sector AOP to the square whose side is a.*

2. Find the area of the sector BOP' bounded by the y-axis, the arc BP' of the hyperbola $y^2 - x^2 = a^2$ (the conjugate of the hyperbola in Ex. 1), and the line OP' drawn from the origin to the point P, P' being the point for which $x = u$. Then show, from the definition above, that $\textit{hysin}^{-1}\frac{u}{a}$ *is* **the ratio** *of twice the sector BOP' to the square whose side is a.*

3. Sketch the curve $y(a^2-x^2)=a^3$. Calculate the area between this curve, the axes, and the ordinate for which $x=u\,(u^2<a^2)$. Show that $\textit{hytan}^{-1}\frac{u}{a}$ *is the ratio of this area to the area of the square whose side is a.*

4. Sketch the curve $y(x^2 - a^2) = a^3$. Calculate the area bounded by this curve, the x-axis, and the ordinate at $x=u\,(u^2>a^2)$. Show that $\textit{hycot}^{-1}\frac{u}{a}$ *is the ratio of this area to the area of the square whose side is a.*

4. Geometrical relations and definitions of the hyperbolic functions. In Fig. 105 P is any point (x, y) on a circle $x^2 + y^2 = a^2$. Let the area of the sector AOP be denoted by u and the angle AOP by θ. Then, by plane trigonometry,

$$u = \tfrac{1}{2}a^2\theta;\ \text{whence,}\ \theta = \frac{2u}{a^2}. \qquad (1)$$

In Fig. 106 P is any point on a rectangular hyperbola $x^2-y^2=a^2$. (The a of the hyperbola bears no relation whatever to the a of

the circle.) Let the area of the sector AOP be denoted by u. Then

$$u = \text{area } OPM - \text{area } APM = \tfrac{1}{2}\,xy - \int_a^x \sqrt{x^2 - a^2}\,dx;$$

whence, $u = \dfrac{a^2}{2}\log\dfrac{x + \sqrt{x^2 - a^2}}{a}$ * $= \dfrac{a^2}{2}\log\dfrac{x+y}{a}$.† (2)

$$\left.\begin{array}{lll} \text{From (2),} & \log\dfrac{x+y}{a} = \dfrac{2u}{a^2}; \text{ whence,} & \dfrac{x+y}{a} = e^{\frac{2u}{a^2}}. \\ \text{Also, since} & x^2 - y^2 = a^2, & \dfrac{x-y}{a} = e^{-\frac{2u}{a^2}}. \end{array}\right\} \quad (3)$$

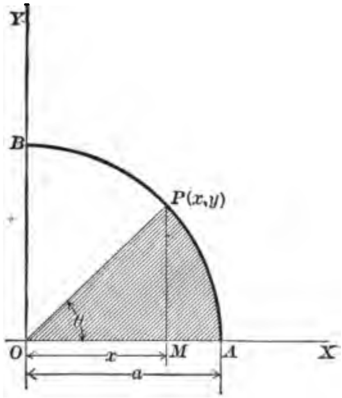

FIG. 105.

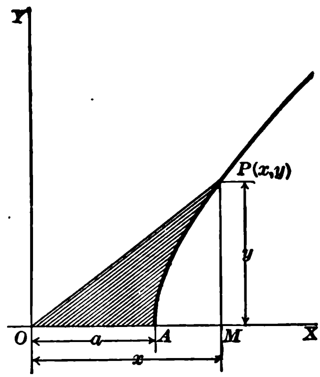

FIG. 106.

From equations (3), on addition and subtraction,

$$\frac{x}{a} = \tfrac{1}{2}\left(e^{\frac{2u}{a^2}} + e^{-\frac{2u}{a^2}}\right); \quad \frac{y}{a} = \tfrac{1}{2}\left(e^{\frac{2u}{a^2}} - e^{-\frac{2u}{a^2}}\right); \quad \therefore \quad \frac{y}{x} = \frac{e^{\frac{2u}{a^2}} - e^{-\frac{2u}{a^2}}}{e^{\frac{2u}{a^2}} + e^{-\frac{2u}{a^2}}}. \quad (4)$$

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* That is, $u = \frac{1}{2}a^2 \operatorname{hycos}^{-1}\frac{x}{a}$; whence, $\operatorname{hycos}^{-1}\frac{x}{a} = \frac{2u}{a^2}$.

† If $a = 1$, $\log(x + y) = 2u =$ twice area AOP. On account of the relation between natural logarithms (*i.e.* logarithms to base e) and the areas of hyperbolic sectors, natural logarithms came to be called *hyperbolic logarithms.* The connection between these logarithms and sectors was discovered by Gregory St. Vincent (1584–1667) in 1647.

Relations (4) lead to **geometrical definitions** of the hyperbolic functions. These definitions are given in the following scheme. This scheme, supplemented by relation (1), also shows the close *geometrical analogies* existing between the hyperbolic and the circular functions.

N.B. In Figs. 105, 106 the a and u of the circle are not related in any way to the a and u of the hyperbola.

In a circle $x^2+y^2=a^2$ (Fig. 105), if P is any point (x, y) and $u =$ area sector AOP,

then
$$\frac{y}{a}=\sin\frac{2u}{a^2},$$
$$\frac{x}{a}=\cos\frac{2u}{a^2},$$
$$\frac{y}{x}=\tan\frac{2u}{a^2};$$
whence,
$$\frac{2u}{a^2}=\sin^{-1}\frac{y}{a}=\cos^{-1}\frac{x}{a}=\tan^{-1}\frac{y}{x}.$$

In a hyperbola $x^2-y^2=a^2$ (Fig. 106), if P is any point (x, y) and $u =$ area sector AOP,

then
$$\frac{y}{a}=\mathbf{hysin}\,\frac{2u}{a^2},$$
$$\frac{x}{a}=\mathbf{hycos}\,\frac{2u}{a^2},$$
$$\frac{y}{x}=\mathbf{hytan}\,\frac{2u}{a^2};$$
whence,
$$\frac{2u}{a^2}=\mathbf{hysin}^{-1}\frac{y}{a}=\mathbf{hycos}^{-1}\frac{x}{a}=\mathbf{hytan}^{-1}\frac{y}{x}.$$

These results may be expressed in words:

The **circular functions** *may be defined by means of the relations connecting a point (x, y) on the circle $x^2+y^2=a^2$ and a certain corresponding circular sector; and the* **hyperbolic functions** *may be defined by means of the relations connecting a point (x, y) on the rectangular hyperbola $x^2-y^2=a^2$ and a certain corresponding hyperbolic sector.*

Each of the **inverse circular functions** *may be expressed as the* **ratio** *of twice the area of a certain sector of a circle of radius a to the square described on the radius of the circle, and each of the* **inverse hyperbolic functions** *may be expressed as the* **ratio** *of twice the area of a certain sector of a rectangular hyperbola of semi-axis a to the square described on this semi-axis.*

(For a more general notion see Ex. 3 following.)

The term *hyperbolic* arose out of the connection of these functions with the hyperbola.

EXAMPLES.

1. Show that $\text{hysin}^{-1}\frac{4}{3} = \text{hycos}^{-1}\frac{5}{3} = \text{hytan}^{-1}\frac{4}{5}$. Represent each of these functions geometrically. Compute $\text{hysin}^{-1}\frac{4}{3}$. [*Ans.* 1.099.]

2. Show that $\text{hysin}^{-1}\frac{3}{4} = \text{hycos}^{-1}\frac{5}{4} = \text{hytan}^{-1}\frac{3}{5}$. Represent each of these functions geometrically. Compute $\text{hysin}^{-1}\frac{3}{4}$. [*Ans.* .693.]

3. Show that, if AP (Fig. 105) is an arc of an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, and u denote the area of the elliptic sector AOP, it is possible to write

$$\frac{x}{a} = \cos\frac{2u}{ab},\quad \frac{y}{b} = \sin\frac{2u}{ab}.$$

Also show that, if AP (Fig. 106) is an arc of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and u denote the area of the hyperbolic sector AOP, then

$$u = \frac{ab}{2}\log\left(\frac{x}{a} + \frac{y}{b}\right);$$

and thence show that

$$\frac{x}{a} = \text{hycos}\,\frac{2u}{ab},\qquad \frac{y}{b} = \text{hysin}\,\frac{2u}{ab}.$$

(Williamson, *Integral Calculus*, Arts. 130, 130 *a*.)

4. Show that a point $P(x, y)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in Ex. 3 may be represented as $(a\cos\theta,\ b\sin\theta)$, and show that θ(= eccentric angle of P) $= (2 \text{ area sector } AOP \div ab)$.

Show that a point $P(x, y)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in Ex. 3 may be represented as $(a\,\text{hycos}\,v,\ b\,\text{hysin}\,v)$, and show that $v = (2 \text{ area sector } AOP \div ab)$.

5. The Gudermannian. Suppose that

$$\sec\phi + \tan\phi = \text{hycos}\,v + \text{hysin}\,v. \tag{1}$$

From (1) and the identities $\sec^2\phi - \tan^2\phi = 1$, $\text{hycos}^2 v - \text{hysin}^2 v = 1$, it follows that

$$\sec\phi = \text{hycos}\,v, \quad (2) \qquad \tan\phi = \text{hysin}\,v. \tag{3}$$

Since [see Art. 2, Ex. 2 (*a*)] $\log(\text{hycos}\,v + \text{hysin}\,v) = v$, relation (1) may be written

$$v = \log(\sec\phi + \tan\phi); \tag{4}$$

that is, by trigonometry,

$$v = \log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = 2.302585\log_{10}\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right). \tag{5}$$

When any one of the relations (1)–(5) *holds between two numbers* v and ϕ, ϕ *is said to be the Gudermannian of* v.* This is expressed by this notation:

$$\phi = gd\, v. \tag{6}$$

In accordance with the usual style of inverse notation each of the relations (4), (5), (6) is expressed

$$v = gd^{-1}\phi. \tag{7}$$

The second members of (4) and (5) are more frequently denoted by the symbol $\lambda(\phi)$, which is read "lambda ϕ," than by $gd^{-1}\phi$.

Geometrical representation of $\lambda(\phi)$ *or* $gd^{-1}\phi$. If at $P(x, y)$ in Fig. 106, $x = a \sec\phi$, then $y = a\tan\phi$, since $x^2 - y^2 = a^2$. On making this substitution for x and y, it can be deduced that

$$\text{area sector } AOP = \tfrac{1}{2} a^2 \log(\sec\phi + \tan\phi). \tag{8}$$

From this,

$$\log(\sec\phi + \tan\phi),\ i.e.\ \lambda(\phi)\ \ (\text{or } gd^{-1}\phi) = \frac{2\cdot\text{sector } AOP}{a^2}. \tag{9}$$

$$\text{From (4), (6), (8),}\quad \phi = gd\left(\frac{2\cdot\text{sector } AOP}{a^2}\right). \tag{10}$$

If the area of sector AOP be denoted by u, relations (9) and (10) may be expressed

$$gd^{-1}\phi = \frac{2u}{a^2},\quad \phi = gd\frac{2u}{a^2}.$$

To construct an angle whose radian measure is ϕ. In Fig. 106, about O as a centre with a radius a describe a circle. From M draw a tangent to this circle, and let the point of contact be at P' in the first quadrant; and draw OP'. Now $OM = OP' \sec MOP'$; *i.e.* $x = a\sec MOP'$. But, according to the hypothesis in the last paragraph, $x = a\sec\phi$. Hence, **angle** $MOP' = \phi$.

If a point $P(x, y)$ on the hyperbola $x^2 - y^2 = a^2$ (see Ex. 4, Art. 4) be denoted as $(a\sec\phi,\ a\tan\phi)$, ϕ is the angle which has just now been constructed.

* This name was given by the great English mathematician Arthur Cayley (1821–1895) "in honour of the German mathematician Gudermann (1798–1852), to whom the introduction of the hyperbolic functions into modern analytical practice is largely due." (Chrystal, *Algebra*, Vol. II., page 288.)

EXAMPLES.

1. Derive result (8).

2. (*a*) Show that, ϕ and v being as in equations (1)-(7),

$$gd\,v = \sec^{-1}(\text{hycos}\,v) = \tan^{-1}(\text{hysin}\,v) = \cos^{-1}(\text{hysec}\,v) = \sin^{-1}(\text{hytan}\,v)$$
$$= \cot^{-1}(\text{hycosec}\,v) = \text{cosec}^{-1}(\text{hycot}\,v);\ \textbf{hytan}\,\tfrac{1}{2}v = \tan\tfrac{1}{2}\phi.$$

(*b*) Show that $gd^{-1}\phi = \text{hycos}^{-1}(\sec\phi) = \text{hysin}^{-1}(\tan\phi)$; $gd\,x = 2\tan^{-1}e^x - \frac{\pi}{2}$.

3. (*a*) Show that the derivative of $\lambda(\phi)$ (*i.e.* $gd^{-1}\phi$) is $\sec\phi$. (*b*) Show that $\lambda(-\phi) = -\lambda(\phi)$. [Suggestion. Show that $\lambda(-\phi) + \lambda(\phi) = \log 1$.] (*c*) Sketch the graph of $\lambda(\phi)$.

4. Show that $\int \text{hysec}\,u\,du = gd\,u$; $\int \sec u\,du = gd^{-1}u$.

Note. **References for collateral reading** *on hyperbolic functions.* Gibson, *Calculus*, §§ 66, 111, 116; Lamb, *Calculus*, Arts. 19, 23, 40, 44, 72, 98, Exs. 2, 3; F. G. Taylor, *Calculus*, Arts. 62–80, 439; W. B. Smith, *Infinitesimal Analysis*, Vol. I., Arts. 99–113; McMahon and Snyder, *Diff. Cal.*, pp. 320–325. For further information see Chrystal, *Algebra* (ed. 1889), Vol. II., Chap. XXIX., §§ 24-31 (pages 276–291); the notes on pages 288, 289 contain interesting information about the history and literature of the subject. Also see Hobson, *Treatise on Plane Trigonometry*, Chap. XVI. An excellent account of hyperbolic functions, starting from the geometrical standpoint and showing practical applications, is given in McMahon, *Hyperbolic Functions* (*i.e.* Merriman and Woodward, *Higher Mathematics*, Chap. IV., pages 107–168).

NOTE B.

INTRINSIC EQUATION OF A CURVE.

1. The intrinsic equation of a curve. Usually the equation of a curve involves either the Cartesian coördinates x and y or the polar coördinates r and θ. In some cases the intrinsic equation is especially useful. In the *intrinsic equation* of a curve the coördinates chosen for any point P are (*a*) *the distance* of P from a chosen fixed point on the curve, this distance being measured *along the curve,* and (*b*) *the angle* made by the tangent at P with a chosen fixed tangent of the curve. These coördinates are denoted respectively by s and ϕ. The relation connecting them, $f(s, \phi) = 0$ say, is called the intrinsic equation of the curve. The term *intrinsic* is used because the coördinates s and ϕ are independent of all points or lines of reference other than the points and tangents of the curve itself.

EXAMPLES.

1. Derive the intrinsic equation of a straight line. Let AB be any straight line. Let O be the chosen fixed point, and $P(s, \phi)$ be any point on the line. It is required to find the equation which is satisfied by s and ϕ.

The direction of the line at P is the same as the direction at O; hence the intrinsic equation is $\phi = 0$.

2. Derive the intrinsic equation of a circle of radius a. Take (Fig. 107) O for the fixed point, and the tangent at O for the tangent of reference. Let $P(s, \phi)$ be any point on the circle. Then $s = \text{arc } OP$ and $\phi = \text{angle } TRP$. Now arc OP $= a \cdot \text{angle } \phi$; *i.e.* $s = a\phi$.

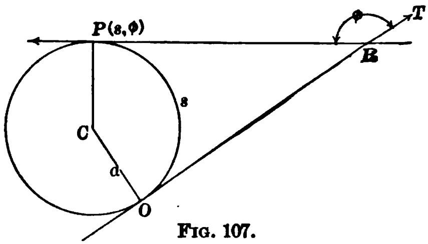

FIG. 107.

2. Derivation of the intrinsic equation of a curve. The intrinsic equation of a curve is usually derived from its equation in Cartesian coördinates or from its equation in polar coördinates. The general method of doing this will now be shown.

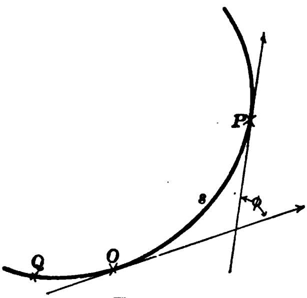

FIG. 108.

Let the equation of the curve be

$$f(x, y) = 0. \tag{1}$$

Take Q for the fixed point, and the tangent at O for the tangent of reference. Take any point P on the curve; let its Cartesian coördinates be x, y, and its intrinsic coördinates be s, ϕ.

Express s in terms of x, y; suppose that

$$s = f_1(x, y). \tag{2}$$

Also express ϕ in terms of x, y; suppose that

$$\phi = f_2(x, y). \tag{3}$$

The elimination of x and y between equations (1), (2), (3), will give the required equation between s and ϕ.

Similarly, let the polar coördinates of P be r and θ, and let the polar equation of the curve be

$$F(r, \theta) = 0. \tag{4}$$

Express s in terms of r, θ; suppose that

$$s = F_1(r, \theta). \tag{5}$$

Also express ϕ in terms of r, θ; suppose that

$$\phi = F_2(r, \theta). \tag{6}$$

The elimination of r and θ between equations (4), (5), (6), will give the required equation between s and ϕ.

NOTE. A tangent parallel to the x-axis is usually chosen for the tangent of reference.

EXAMPLES.

1. Derive the intrinsic equation of the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \tag{1}$$

Take the cusp on the positive part of the x-axis for the fixed point, and the tangent there for the tangent of reference. Then at any point $P(x, y)$ on the arc in the first quadrant

$$\tan\phi = -(y^{\frac{1}{3}} \div x^{\frac{1}{3}}), \tag{2}$$

and

$$s = \tfrac{3}{2}\, a^{\frac{1}{3}}\,(a^{\frac{2}{3}} - x^{\frac{2}{3}}). \tag{3}$$

From (1) and (2), $\quad \sec^2\phi = \tan^2\phi + 1 = a^{\frac{2}{3}} \div x^{\frac{2}{3}}.$

Substitution for $x^{\frac{2}{3}}$ in (3) gives $2s = 3a\sin^2\phi$.

2. If in Ex. 1 the chosen fixed point O be at a distance b along the curve from the cusp and the chosen fixed tangent (not necessarily at O) make an angle α with the tangent at the cusp, show that the intrinsic equation of the hypocycloid is

$$2(s + b) = 3a\sin^2(\phi + \alpha).$$

3. Find the intrinsic equation of the cardioid $r = a(1 - \cos\theta)$.

Let the polar origin be chosen for the fixed point, and the tangent there be chosen for the tangent of reference. Let $P(x, y)$ be any point on the cardioid. Then

$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4a\left(1 - \cos\frac{\theta}{2}\right). \tag{1}$$

Also, (Art. 60), $\phi = \theta + \tan^{-1}\frac{r\,d\theta}{dr} = \theta + \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{3}{2}\theta.$ (2)

On substituting in (1) the value of θ from (2),

$$s = 4a\left(1 - \cos\frac{\phi}{3}\right).$$

4. If in Ex. 3 the chosen fixed point be at a distance b from the polar origin and the chosen tangent of reference make an angle α with the tangent at the polar origin, show that the intrinsic equation of the cardioid is

$$s + b = 4a\left(1 - \cos\frac{\phi + \alpha}{3}\right).$$

5. Derive the intrinsic equation of each of the following curves, the fixed point and the fixed tangent being as indicated: (1) the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, the vertex and tangent thereat; (2) the parabola $y^2 = 4ax$, the vertex and tangent thereat; (3) the parabola $r = a\sec^2\frac{\theta}{2}$, as in (2); (4) the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, with reference to (a) the origin and tangent thereat, (b) the vertex and tangent thereat; (5) the logarithmic spiral $r = ce^{a\theta}$; (6) the semi-cubical parabola $3ay^2 = 2x^3$, the origin and tangent thereat; (7) the curve $y = a\log\sec\frac{x}{a}$, the origin; (8) the semi-cubical parabola $y^3 = ax^2$; (9) the tractrix $x = \sqrt{c^2 - y^2} + c\log\frac{c + \sqrt{c^2 - y^2}}{y}$, the point (0, c). (For an account of the tractrix and for various problems which reveal its properties, see the text-books of Williamson, Byerly, Lamb, and F. G. Taylor, on the calculus.)

[*Answers:* Ex. 5. (1) $s = a\tan\phi$, (2) $s = a\tan\phi\sec\phi + a\log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)$, (3) as in (2), (4) ($a$) $s = 4a(1 - \cos\phi)$, (b) $s = 4a\sin\phi$, (5) $s = c(e^{a\phi} - 1)$, (6) $9s = 4a(\sec^3\phi - 1)$, (7) $s = a\log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)$, (8) $27s = 8a(\sec^3\phi - 1)$, (9) $s = c\log\sec\phi$.]

3. Radius of curvature derived from the intrinsic equation. The radius of curvature at a point on a curve can very easily be deduced from the intrinsic equation. For, according to Arts. 146, 147, the radius of curvature being denoted by R,

$$R = \frac{ds}{d\phi}.$$

EXAMPLES.

1. In Art. 2, Ex. 5 (1), $R = a\sec^2\phi$.

2. Find the radius of curvature for each of the curves in Art. 2, Ex. 1, Ex. 3, Ex. 5 (4), (5), (6), (9).

[*Answers:* Ex. 1. $\frac{3}{2}a\sin 2\phi$; Ex. 3. $\frac{4}{3}a\sin\frac{\phi}{3}$; Ex. 5 (4). ($a$) $4a\sin\phi$, (b) $4a\cos\phi$; (5) $ace^{a\phi}$; (6) $\frac{4}{3}a\sec^3\phi\tan\phi$; (9) $c\tan\phi$.]

Note. On *the intrinsic equation of a curve*, see Todhunter, *Integral Calculus*, Arts. 103–119; Byerly, *Integral Calculus*, Arts. 114–123.

NOTE C.

EVALUATION OF INDETERMINATE FORMS.

1. Indeterminate forms. When $x = 1$, the value of the fraction $\frac{x^2-4}{x-2}$ is 3; when $x = 1.5$, its value is 3.5; when $x = 1.9$, its value is 3.9; *when* $x = 2$, *the fraction takes the form* $\frac{0}{0}$; when $x = 2.1$, the value of the fraction is 4.1; when $x = 2.5$, its value is 4.5. Thus the fraction has definite values when $x = \cdots, 1, \cdots, 1.5, \cdots, 1.9, 2.1, \cdots, 2.5, \cdots$. It is reasonable to conclude that *it has a definite value when* $x = 2$. Put $x = 2 + h$; then the fraction becomes $\frac{(2+h)^2-4}{2+h-2}$, *i.e.* $4 + h$. The limiting value of this when h approaches zero, is 4. Accordingly, $\lim_{x \doteq 2}\frac{x^2-4}{x-2} = 4$. Thus the true, or limiting, value of the form $\frac{0}{0}$ which appeared above is 4.

The form $\frac{0}{0}$ is usually called *an indeterminate form*. This name is a misnomer; for, as will presently appear, the value of such a form may be determined. A better name, perhaps, is *an undetermined form*, or *an elusive form*, or *an illusory form*. The evaluation of expressions taking this form can be effected in various ways. Several of these ways are shown in text-books on algebra, and will not be discussed here;* this Note is concerned only with *the evaluation of illusory forms by means of the calculus.*

NOTE 1. The only applications of this Note in the preceding part of this book are in Art. 165.

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* In text-books on Algebra the form $0 \div 0$ is often called a *vanishing fraction;* for its evaluation, see (among others) Hall and Knight, *Algebra* (4th edition), §§ 271, 272.

There are various illusory forms besides $\frac{0}{0}$; viz. $\frac{\infty}{\infty}$, 1^∞, 0^0, ∞^0, $\infty \cdot 0$, $\infty - \infty$. Their evaluation will be found to depend upon the evaluation of $0 \div 0$.

NOTE 2. The chief methods used in algebra for evaluating expressions having the form $0 \div 0$, are :

(*a*) By removing common factors from the numerator and denominator. Thus $\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)\times 1} = x+2$; this, when $x = 2$, is equal to 4.

(*b*) By expanding in series. For example, $\sin x \div x$ takes the form $0 \div 0$ when $x = 0$. On using the expansion for $\sin x$ given in Ex. 2, Art. 178, it is easily seen that $\lim_{x \doteq 0} \frac{\sin x}{x} = 1$.

NOTE 3. Illusory forms have frequently appeared in this book; for instances, see Art. 14 (Exs. 11–14), Art. 22, and Chapter IV.

2. Evaluation of expressions taking the form $\frac{0}{0}$. Suppose that $f(x)$ and $\phi(x)$ are continuous functions of x, and that $f(a) = 0$ and $\phi(a) = 0$. Suppose further that $f(x+h)$ and $\phi(x+h)$ can be expanded by Taylor's formula in the neighbourhood of $x = a$. Let it be required to determine the true value of $\frac{f(a)}{\phi(a)}$.

Now, **the value of** $\frac{f(a)}{\phi(a)} = \lim_{h \doteq 0} \frac{f(a+h)}{\phi(a+h)}$. (1)

[In what follows, the expansion of $f(a+h)$ is obtained by writing the expansion of $f(x+h)$ and then substituting a for x therein.]

By Taylor's theorem [Art. 176, formula (9)],

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a+\theta_1 h)}{\phi(a) + h\phi'(a+\theta_2 h)} = \frac{f'(a+\theta_1 h)}{\phi'(a+\theta_2 h)}.$$

(The θ's are each less than 1.)

Hence, by (1), the value of $\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}$.

If $f'(a)$ and $\phi'(a)$ are both zero, then

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta_3 h)}{\phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a+\theta_4 h)} = \frac{f''(a+\theta_3 h)}{\phi''(a+\theta_4 h)}.$$

Hence, by (1), the value of $\frac{f(a)}{\phi(a)} = \frac{f''(a)}{\phi''(a)}$.

On proceeding in this way it can be shown, by means of Taylor's theorem, that, **if**, *for $x = a$, $f(x)$ and $\phi(x)$ and all their derivatives up to and including*

their nth derivatives, are zero, while $f^{(n+1)}(a)$ and $\phi^{(n+1)}(a)$, are not both zero, **then**

$$\textbf{the value of } \frac{f(a)}{\phi(a)}=\frac{f^{(n+1)}(a)}{\phi^{(n+1)}(a)}. \tag{2}$$

Result (2) may also be expressed thus:

$$\lim_{x\doteq a}\frac{f(x)}{\phi(x)}=\lim_{x\doteq a}\frac{f^{(n+1)}(x)}{\phi^{(n+1)}(x)}.$$

EXAMPLES.

1. Evaluate $\frac{x^2-4}{x-2}$ when $x=2$. (See Art. 1.)

$$\text{Value}_{x\doteq 2}\frac{x^2-4}{x-2}=\text{value}_{x\doteq 2}\frac{D(x^2-4)}{D(x-2)}=\text{value}_{x\doteq 2}\frac{2x}{1}=4.$$

2. Evaluate $(x-\sin x)\div x^3$ when $x=0$. In this case,

$$\lim_{x\doteq 0}\frac{x-\sin x}{x^3}=\lim_{x\doteq 0}\frac{1-\cos x^*}{3x^2}=\lim_{x\doteq 0}\frac{\sin x^*}{6x}=\lim_{x\doteq 0}\frac{\cos x}{6}=\frac{1}{6}.$$

Note. The labour of evaluating $f(a)\div\phi(a)$ may be lightened in the following cases:

(*a*) If, in the course of the reduction a factor, say $\psi(x)$, appears in both the numerator and the denominator, this common factor may be cancelled. [Compare Art. 1, Note 2 (*a*).]

(*b*) If at any stage during the process of evaluation a factor, say $\psi(x)$, appears only in the numerator or only in the denominator, and $\psi(a)$ is not zero, the value of $\psi(a)$ may be substituted immediately for $\psi(x)$. This will lessen the labour in the succeeding differentiations.

3. Evaluate the following: (1) $\frac{a^x-b^x}{x}$, when $x=0$; (2) $\frac{\sin^{-1}x}{x}$, when $x=0$; (3) $\frac{x^n-a^n}{x-a}$ when $x=a$; (4) $\frac{e^x-e^{-x}}{\sin x}$, when $x=0$; (5) $\frac{1-\cos z}{z^2}$, when $z=0$.

4. Find the following:

(1) $\lim_{x\doteq 0}\frac{(x-5)^2\sin x}{x}$; (2) $\lim_{x\doteq 2}\frac{(x-5)^2\log(3-x)}{\sin(x-2)}$;

(3) $\lim_{x\doteq 0}\frac{e^x+e^{-x}+2\cos x-4}{x^4}$; (4) $\lim_{x\doteq 0}\frac{\tan x-\sin x}{x-\sin x}$;

(5) $\lim_{x\doteq 0}\frac{1-\cos x}{\cos x\sin^2 x}$.

[*Answers:* Exs. 3. $\log\frac{a}{b}$, 1, na^{n-1}, 2, $\frac{1}{2}$; Exs. 4. 25, -9, $\frac{1}{6}$, 3, $\frac{1}{2}$.]

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* Which is in the form $0\div 0$.

3. Evaluation of expressions taking the form $\frac{\infty}{\infty}$. Suppose that $f(a) = \infty$ and $\phi(a) = \infty$, and let it be required to determine the value of $\frac{f(a)}{\phi(a)}$.

Now $\frac{f(a)}{\phi(a)} = \frac{\frac{1}{\phi(a)}}{\frac{1}{f(a)}}$. The latter is in the form $\frac{0}{0}$. Application of result (2) Art. 2, gives

$$\text{value of } \frac{f(a)}{\phi(a)} = \frac{-\frac{\phi'(a)}{[\phi(a)]^2}}{-\frac{f'(a)}{[f(a)]^2}} = \left(\text{value of } \frac{f(a)}{\phi(a)}\right)^2 \cdot \frac{\phi'(a)}{f'(a)}. \tag{1}$$

From (1), value of $\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}$; *i.e.* $\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \lim_{x \doteq a} \frac{f'(x)}{\phi'(x)}$.

Similarly, if $\frac{f'(a)}{\phi'(a)}$ is also illusory, its value can be shown to be $\frac{f''(a)}{\phi''(a)}$; and so on. It thus appears that the methods for evaluating the illusory forms in Arts. 2 and 3 are *the same*.

EXAMPLES.

1. Evaluate $\frac{x}{\log x}$ when $x = \infty$. (See Art. 8, Note 2.)

$$\lim_{x \doteq \infty} \frac{x}{\log x} = \lim_{x \doteq \infty} \frac{1}{\frac{1}{x}} = \lim_{x \doteq \infty} x = \infty.$$

2. Evaluate $\frac{x}{e^x}$, $\frac{x^2}{e^x}$, $\frac{x^n}{e^x}$, when $x = \infty$.

3. Find: (1) $\lim_{x \doteq 0} \frac{\log x}{\cot x}$; (2) $\lim_{x \doteq \frac{\pi}{2}} \frac{\sec x}{\sec 3x}$; (3) $\lim_{x \doteq \frac{\pi}{2}} \frac{\tan 5x}{\tan x}$.

[*Answers:* Exs. 2. 0, 0, 0; Exs. 3. 0, -3, $\frac{1}{5}$.]

4. Evaluation of other indeterminate forms. The evaluation of these forms can be made to depend on Arts. 2, 3.

(*a*) **The form $0 \cdot \infty$.** Suppose that $\phi(a) = 0$ and $\psi(a) = \infty$, and let the value of $\phi(a) \cdot \psi(a)$ be required.

Now $\phi(a) \cdot \psi(a) = \phi(a) \div \frac{1}{\psi(a)}$, which is in the form $\frac{0}{0}$; also $\phi(a) \cdot \psi(a) = \psi(a) \div \frac{1}{\phi(a)}$, which is in the form $\frac{\infty}{\infty}$. Thus expressions having the form $0 \cdot \infty$ can be transformed into expressions having the form $0 \div 0$ or $\infty \div \infty$.

EXAMPLES.

1. $\text{Lim}_{x\doteq 0}(x\cdot\cot x)=\lim_{x\doteq 0}\dfrac{x}{\tan x}\left(i.e.\ \dfrac{0}{0}\right)=\lim_{x\doteq 0}\dfrac{1}{\sec^2 x}=1.$

2. Determine: (1) $\lim_{x\doteq\frac{\pi}{2}}\left(\dfrac{\pi}{2}-x\right)\tan x$; (2) $\lim_{x\doteq\infty} a^x\cdot\sin\dfrac{m}{a^x}$; (3) $\lim_{x\doteq 1}(x-1)\tan\dfrac{\pi x}{2}$. $\left[\textit{Answers: } 1,\ m,\ -\dfrac{2}{\pi}.\right]$

(*b*) **The form $\infty-\infty$.** By combining terms and simplifying, an expression having the form $\infty-\infty$ may be reduced to a definite value, or to one of the preceding illusory forms.

EXAMPLES.

3. $\text{Lim}_{x\doteq 2}\left(\dfrac{2}{x^2-4}-\dfrac{1}{x-2}\right)=\lim_{x\doteq 2}\dfrac{2x-x^2}{x^2-4}=\lim_{x\doteq 2}\dfrac{2-2x}{2x}=-\dfrac{1}{2}.$

4. Find: $\lim_{x=1}\left(\dfrac{x}{x-1}-\dfrac{1}{\log x}\right)$, $\lim_{x\doteq 0}\left\{\dfrac{1}{x}-\dfrac{1}{x^2}\log(1+x)\right\}$, $\lim_{x\doteq\infty}(x-\sqrt{x^2-a^2})$. [*Answers:* $\frac{1}{2}$, $\frac{1}{2}$, 0.]

(*c*) **The forms 1^∞, ∞^0, 0^0.** Suppose that $[\phi(a)]^{\psi(a)}$ is in one of these forms. Put $u=[\phi(a)]^{\psi(a)}$; then $\log u=\psi(a)\cdot\log[\phi(a)]$.

The second member is in the form $0\cdot\infty$; and, accordingly, $\log u$ may be determined. Then the value of u can be deduced from the value of $\log u$.

EXAMPLES.

5. Evaluate $(1-x)^{\frac{1}{x}}$ when $x=0$. (The form then is 1^∞.)

Put $\quad u=(1-x)^{\frac{1}{x}};\quad$ then $\log u=\dfrac{\log(1-x)}{x}.$

Accordingly, $\lim_{x\doteq 0}\log u=\lim_{x\doteq 0}\left(\dfrac{-1}{1-x}\right)=-1.\quad\therefore\ u=\dfrac{1}{e}$ when $x=0$.

6. Find $\lim_{x\doteq 0}(x^x)$. (This form is 0^0.)

Put $\quad u=x^x;\quad$ then $\log u=x\log x.$

Accordingly,

$$\lim_{x\doteq 0}\log u=\lim_{x\doteq 0}\frac{\log x}{x^{-1}}=\lim_{x\doteq 0}\frac{\frac{1}{x}}{-x^{-2}}=\lim_{x\doteq 0}(-x)=0;$$

consequently, $\quad u=e^0=1$ when $x=0$.

7. Evaluate the following: (1) $\left(1+\dfrac{1}{x}\right)^x$ when $x=\infty$; (2) $\sin x^{\tan x}$ when $x=0$; (3) $x^{\frac{1}{x}}$ when $x=\infty$; (4) $(1-x)^{\frac{1}{x}}$ when $x=\infty$; (5) $\left(1+\dfrac{1}{x}\right)^{x^2}$ when $x=\infty$; (6) $\left(1+\dfrac{1}{x^2}\right)^x$ when $x=\infty$; (7) $x^{\frac{1}{x-1}}$ when $x=1$; (8) $x^{\frac{1}{x-1}}$ when $x=\infty$; (9) $x^{\sin x}$ when $x=0$. [*Answers:* (1) e, (2) 1, (3) 1, (4) 1, (5) ∞, (6) 1, (7) e, (8) 1, (9) 1.]

NOTE. **References for collateral reading** *on illusory forms.* For a fuller discussion on the evaluation of expressions in these forms, and for many examples, see McMahon and Snyder, *Diff. Cal.*, Chap. V., pages 115–131; F. G. Taylor, *Calculus*, Chap. XII., pages 136–148; Echols, *Calculus*, Chap. VII.; also Gibson, *Calculus*, Arts. 161, 162. For a general treatment of the subject see Chrystal, *Algebra*, Vol. II., Chap. XXV.

NOTE D.

APPLICATIONS TO MECHANICS.

N.B. For a full explanation and discussion of the mechanical terms in this note, see text-books on *Mechanics*.

1. Mass, density, centre of mass. For this note the following definition of mass may serve: *The mass of a body is the quantity of matter which the body contains.** The principal standards of mass are two particular platinum bars; the one is the "imperial standard pound avoirdupois," which is kept in London, and the other is the "kilogramme des archives," which is kept in Paris.

NOTE. The *weight* of a body is the measure of the earth's attraction upon the body, and depends both on the mass of the body and its distance from the centre of the earth. The same body, while its mass remains constant, has different weights according to the different positions it takes with respect to the centre of the earth.

The density of a body is the ratio of the measure of its mass to the measure of its volume; that is, the density is the number of units of mass in a unit of volume. *The density at a point* is the limiting value of the ratio of (the measure of) the mass of an infinitesimal volume about the point to (the measure of) the infinitesimal volume. A body is said to be *homogeneous* when the density is the same at all points. If a body is not homogeneous, the "density of a body," defined above, is the *average* or *mean density* of the body.

Centre of mass. Suppose there are particles whose masses are m_1, m_2, m_3, $\cdots$, and whose distances from any plane are, respectively, d_1, d_2, d_3, $\cdots$. Let a number D be calculated such that

$$D = \frac{m_1 d_1 + m_2 d_2 + m_3 d_3 \cdots}{m_1 + m_2 + m_3 + \cdots}; \quad i.e. \text{ let } D = \frac{\Sigma\, md}{\Sigma m}.$$

For any given plane, D evidently has a definite value.

* A real definition of mass, one that is strictly logical and fully satisfactory, is explained in good text-books on dynamics and mechanics. (For example, see MacGregor, *Kinematics and Mechanics*, 2d ed., Art. 289.)

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $\cdots$, respectively, be the coördinates of these particles with respect to three coördinate planes at right angles to one another, then the point $(\bar{x}, \bar{y}, \bar{z})$, such that

$$\bar{x} = \frac{\Sigma\, mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma\, my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma\, mz}{\Sigma m}, \tag{1}$$

is called **the centre of mass** of the set of particles.

If the matter "be distributed continuously" (as along a line, straight or curved, or over a surface, or throughout a volume), and if Δm denote the element of mass about any point (x, y, z), then, on taking all the points into consideration, equations (1) may be written:

$$\bar{x} = \frac{\lim_{\Delta m \doteq 0} \Sigma x \cdot \Delta m}{\lim_{\Delta m \doteq 0} \Sigma\, \Delta m}, \text{ and similarly for } \bar{y} \text{ and } \bar{z}. \tag{2}$$

From (2), by the definition of an integral,

$$\bar{x} = \frac{\int x\, dm}{\int dm}, \quad \bar{y} = \frac{\int y\, dm}{\int dm}, \quad \bar{z} = \frac{\int z\, dm}{\int dm}. \tag{3}$$

If ρ denote density of an infinitesimal dv about a point, then

$$dm = \rho\, dv \quad (4); \quad \text{and, on integration,} \quad m = \int \rho\, dv. \tag{5}$$

Ex. Write formulas (3), putting $\rho\, dv$ for dm.

Suppose that the body is not homogeneous; that is, suppose that the density of the body varies from point to point. Let ρ denote the density at any point (x, y, z), let dv denote an infinitesimal volume about that point, and let $\bar{\rho}$ denote the average or mean density of the body. Then

$$\bar{\rho} = \frac{\text{mass of body}}{\text{vol. of body}} = \frac{\int \rho\, dv}{\int dv}.$$

NOTE. The term *centre of mass* is used also in cases in which matter is supposed to be concentrated along a line or curve, or on a surface. In these cases the terms *line-density* and *surface-density* are used.

EXAMPLES.

1. In a quadrant of a thin elliptical plate whose semi-axes are a and b, the density varies from point to point as the product of the distances of each point from the axes. Find the mass, the mean density, and the position of the centre of mass, of the quadrant.

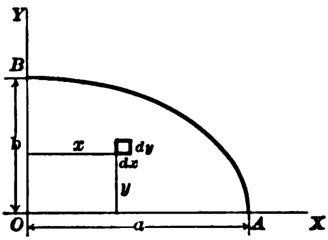

FIG. 109.

Choose rectangular axes as in the figure. At any point $P(x, y)$, let ρ denote the density and dm denote the mass of a rectangular bit of the plate, say, $dx \cdot dy$. Let M denote the mass, $\bar{\rho}$ the mean density, and $(\bar{x}, \bar{y})$ the centre of mass, of the quadrant.

Now $dm = \rho\, dx\, dy$. But $\rho \propto xy$; *i.e.* $\rho = k\,xy$, in which k denotes some constant.

Accordingly, $$M = \int dm = \int_{x=0}^{x=a}\int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} k\,xy\,dy\,dx = \tfrac{1}{8}\,k\,a^2b^2.$$

Also, $$\bar{\rho} = \frac{\text{mass of quadrant}}{\text{volume of quadrant}} = \frac{\frac{1}{8}\,k\,a^2b^2}{\frac{1}{4}\pi ab} = \frac{kab}{2\pi}.$$

Here $$\bar{x} = \frac{\int \rho\cdot x\cdot dv}{\int \rho\cdot dv} = \frac{k\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2y\,dy\,dx}{M} = \frac{\frac{1}{15}\,ka^3b^2}{\frac{1}{8}\,ka^2b^2} = \tfrac{8}{15}\,a.$$

Similarly, $\bar{y} = \frac{8}{15}\,b$. Hence, centre of mass is $(\frac{8}{15}\,a, \frac{8}{15}\,b)$.

2. Find the centre of mass of a solid hemisphere, radius a, in which the density varies as the distance from the diametral plane. Also find the mean density.

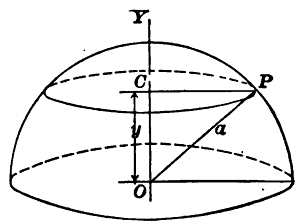

FIG. 110.

Symmetry shows that the centre of mass is in OY.

Take a section parallel to the diametral plane and at a distance y from it.

The area of this section

$$= \pi\cdot\overline{CP}^2 = \pi(a^2 - y^2).$$

For this section, $\rho \propto y$, *i.e.* $\rho = ky$, say.

Then $$\bar{y} = \frac{\int_0^a \rho\cdot y\cdot\pi(a^2-y^2)dy}{\int_0^a \rho\pi(a^2-y^2)dy} = \frac{k\pi\int_0^a y^2(a^2-y^2)dy}{k\pi\int_0^a y(a^2-y^2)dy} = \tfrac{8}{15}\,a.$$

Also $$\bar{\rho} = \frac{M}{\text{vol.}} = \frac{\frac{1}{4}\,k\pi a^4}{\frac{2}{3}\,\pi a^3} = \tfrac{3}{8}\,ka.$$

This is the density at a distance $\frac{3}{8}\,a$ from the diametral plane.

3. The quadrant of a circle of radius a revolves about the tangent at one extremity. Find the position of the mass-centre of the surface thus generated. In this case let the "surface-density" be denoted by ρ. Symmetry shows that the mass-centre is in the line PL. Let y denote the distance of the mass-centre from OX.

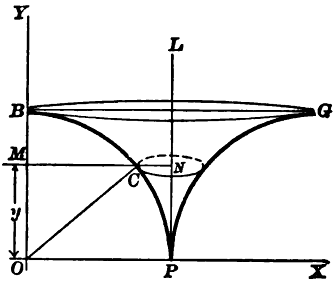

FIG. 111.

In PL take any point N, at a distance y, say, from OX. Through N pass a plane at right angles to PL, and pass another parallel plane at an infinitesimal distance dy from the first plane. These planes intercept an infinitesimal zone, of breadth ds say, on the surface generated.

Area of this zone $= 2\pi \cdot CN \cdot ds = 2\pi(MN - MC)ds$.

Now, at $C(x, y)$ $\qquad x^2 + y^2 = a^2$.

Accordingly, $\quad ds = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2} \cdot dy = \dfrac{a}{\sqrt{a^2 - y^2}} dy.$

Hence, area zone $= 2\pi(a - \sqrt{a^2 - y^2}) \dfrac{a}{\sqrt{a^2 - y^2}} dy = 2\pi a\left(\dfrac{a}{\sqrt{a^2 - y^2}} - 1\right) dy.$

$$\therefore \bar{y} = \frac{\int_{y=0}^{y=a} \rho y \cdot (2\pi \cdot CN \cdot ds)}{\rho \cdot \text{area zone}} = \frac{2\pi a\rho \int_0^a y\left(\dfrac{a}{\sqrt{a^2 - y^2}} - 1\right) dy}{2\pi a\rho \int_0^a \left(\dfrac{a}{\sqrt{a^2 - y^2}} - 1\right) dy}$$

$$= \frac{a}{\pi - 2} = .876\, a.$$

4. In the following lines, curves, surfaces, and solids, find the position of the centre of mass; and, in cases in which the matter is not distributed homogeneously, also find the total mass and the mean density ("line-density," "surface-density," or "density," as the case may be). (The density is uniform, unless otherwise specified.)

(1) A straight line of length l in which the line-density varies as (is k times), (a) the distance from one end; (b) the square of this distance; (c) the square root of this distance.

(2) An arc of a circle, radius r, subtending an angle 2α at the centre.

(3) A fine uniform wire forming three sides of a square of side a.

(4) A quadrantal arc of the four-cusped hypocycloid.

(5) A plane quadrant of an ellipse, semi-axes a and b.

(6) The area bounded by a semicircle of radius r and its diameter, (a) when the surface density is uniform; (b) when the surface density at any point varies as (is k times) its distance from the diameter.

(7) The area bounded by the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the axes.

(8) The cardioid $r = 2a(1 - \cos\theta)$.

(9) A circular sector having radius r and angle 2α.

(10) The segment bounded by the arc of the sector in Ex. (9) and its chord.

(11) The crescent or lune bounded by two circles which touch each other internally, their diameters being d and $\frac{3}{4}d$, respectively.

(12) The curved surface of a right circular cone of height h, (a) when the surface density at a point varies as its distance from a plane which passes through the vertex and is at right angles to the axis of the cone; (b) when the surface density is uniform.

(13) A thin hemispherical shell of radius a, in which the surface density varies as the distance from the plane of the rim.

(14) A right circular cone of height h in which, (a) the density of each infinitely thin cross-section varies as its distance from the vertex; (b) the density is uniform.

(15) Show that the mass-centre of a solid paraboloid generated by revolving a parabola about its axis, is on the axis of revolution at a point two-thirds the distance of the base from the vertex.

(16) A solid hemisphere of radius r, (a) when the density is uniform; (b) when the density varies as the distance from the centre.

(17) Show that the mass-centre of the solid generated by the revolution of the cardioid in Ex. (8) about its axis, is on this axis at a distance $\frac{8}{5}a$ from the cusp.

(18) If the density ρ at a distance r from the centre of the earth is given by the formula $\rho = \rho_0 \dfrac{\sin kr}{kr}$, in which ρ_0 and k are constants, show that the earth's mean density is

$$3\rho_0 \frac{\sin kR - kR\cos kR}{k^3R^3},$$

in which R denotes the earth's radius. (Lamb's *Calculus*.)

[**Answers:** (1) $\frac{2}{3}l$ from that end, $M = \frac{1}{2}kl^2$, $\bar{\rho} = \frac{1}{2}kl$; ($b$) $\frac{3}{4}l$, $M = \frac{1}{3}kl^3$, $\bar{\rho} = \frac{1}{3}kl^2$; ($c$) $\frac{3}{5}l$, $M = \frac{2}{3}kl^{\frac{3}{2}}$, $\bar{\rho} = \frac{2}{3}kl^{\frac{1}{2}}$. (2) On radius bisecting the arc at distance $r\dfrac{\sin\alpha}{\alpha}$ from centre. (3) At a distance $\frac{1}{6}a$ from the centre of the square. (4) Point distant $\frac{2}{5}a$ from each axis. (5) Point distant $\dfrac{4a}{3\pi}$ from axis $2a$, $\dfrac{4b}{3\pi}$ from axis $2b$. (6) (a) On middle radius, at point distant $\dfrac{4a}{3\pi}$ from the diameter; (b) On middle radius, at point $.589a$ from the diameter, mean density $= .4244$ maximum density. (7) Point distant $\frac{1}{5}a$ from each axis. (8) The point $(\pi, \frac{5}{3}a)$. (9) On middle radius of sector, at distance $\frac{2}{3}r\dfrac{\sin\alpha}{\alpha}$ from the centre. (10) On the bisector of the chord, at distance

$\frac{2}{3}r\dfrac{\sin^3\alpha}{\alpha-\sin\alpha\cos}$ from the centre. (11) On the diameter through the point of contact and distant $\frac{19}{30}d$ from that point. (12) (a) On the axis, at distance $\frac{3}{4}h$ from the vertex; (b) on axis, at distance $\frac{2}{3}h$ from vertex. (13) On the radius perpendicular to the plane of the rim, at a distance $\frac{2}{3}a$ from the centre. (14) (a) On the axis, $\frac{4}{5}h$ from the vertex; the mean density is the same as the density at the cross-section distant $\frac{3}{4}h$ from the vertex; (b) on the axis, at a distance $\frac{3}{4}h$ from the vertex. (16) (a) On a radius perpendicular to the base, at a distance $.375\,r$ from it; (b) on radius as in (a), at distance $.4\,r$ from the base.]

2. Moment of inertia. Radius of gyration. These quantities are of immense importance in mechanics and its practical applications.

Moment of inertia. Let there be a set of particles whose masses are, respectively, $m_1, m_2, m_3, \cdots$, and whose distances from a chosen fixed line are, respectively, $r_1, r_2, r_3, \cdots$. The quantity

$$m_1r_1^2+m_2r_2^2+m_3r_3^2+\cdots, \text{ i.e. } \Sigma\, mr^2 \qquad (1)$$

is called **the moment of inertia** of the set of particles with respect to the fixed line, or *axis*, as it is often called. It is evident that for any chosen line and system of particles the moment of inertia has a definite value. In what follows, the moment of inertia will be denoted by I.

It can be shown, by the same reasoning as in Art. (1), that definition (1) can be extended to the case of any continuous distribution of matter (whether along a line or curve, or over a surface, or throughout a solid) and any chosen axis; thus,

$$I=\int r^2\,dm,$$

in which r denotes the distance of any point from the axis, and dm an infinitesimal element of mass about that point.

Radius of gyration. In the case of any distribution of matter and a fixed line, or axis, the number k, which is such that

$$k^2=\frac{\text{the moment of inertia}}{\text{the mass}}=\frac{\int r^2\,dm}{\int dm},$$

is called the (length of the) **radius of gyration** about that axis.

EXAMPLES.

1. Find the radius of gyration about its line of symmetry of an isosceles triangle of base $2a$ and altitude h.

The density per unit of area will be denoted by ρ.

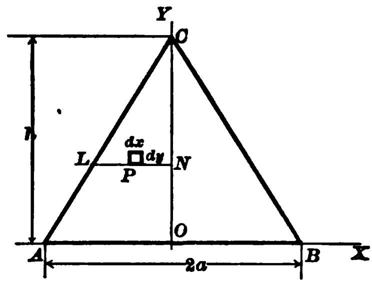

FIG. 112.

Let P be any point in the triangle, and make the construction shown in the figure. Denote NO by y.

Then $$k^2 = \frac{2\,\Sigma\,\overline{PN}^2 \cdot \rho \cdot dx\,dy \text{ over } AOC}{\Sigma\rho \cdot dx\,dy \text{ over } ABC} = \frac{2\rho\int_{y=0}^{y=h}\int_{x=0}^{x=LN} x^2 \cdot dx\,dy}{\rho\,ah}.$$

Now $$\frac{LN}{AO} = \frac{CN}{CO}, \text{ i.e. } \frac{LN}{a} = \frac{h-y}{h}; \text{ whence } LN = \frac{a}{h}(h-y).$$

$$\therefore\ k^2 = \frac{\frac{1}{6}a^3h}{ah} = \frac{1}{6}a^2; \text{ whence } k = \frac{a}{\sqrt{6}}.$$

In this example, the moment of inertia is $\frac{1}{6}a^3h$.

2. Show that the moment of inertia of a homogeneous thin circular plate about an axis through its centre and perpendicular to its plane is $\frac{1}{2}\rho\pi a^4$, in which ρ denotes the surface density, and that its radius of gyration is $\frac{1}{2}a\sqrt{2}$.

$$\left[\text{On using polar coördinates, } I = \int r^2 \cdot dm = \int r^2 \cdot \rho \cdot dA = \rho\int_0^{2\pi}\int_0^a r^2 \cdot r\,dr\,d\theta.\right]$$

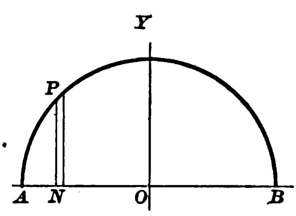

FIG. 113.

3. Find the moment of inertia of a solid homogeneous sphere of radius a about a diameter, m being the mass per unit of volume. Suppose that the sphere is generated by the revolution of the semicircle APB about the diameter AB. Let rectangular axes be chosen as in the figure. At any point $P(x, y)$ on the semicircle take a thin rectangular strip PN at right angles to AB

and having a width Δx. This strip, on the revolution, generates a thin circular plate. It follows from Ex. 2, since m is the mass per unit of volume, that

$$I \text{ of this plate about } AB = \frac{m}{2}\pi \cdot \overline{PN}^4 \cdot \Delta x.$$

$$\therefore I \text{ of sphere} = \Sigma \frac{m}{2}\pi \cdot \overline{PN}^4 \Delta x \text{ from } A \text{ to } B$$

$$= 2 \cdot \frac{m\pi}{2} \int_0^a (a^2 - x^2)^2 dx = \tfrac{8}{15} m\pi a^5.$$

Here, on denoting the mass of the sphere by M,

$$M = \tfrac{4}{3} m\pi a^3;$$

hence, $I = \tfrac{2}{5} Ma^2;$

accordingly, $k^2 = \tfrac{2}{5} a^2;$

and thus, $k = .632\, a.$

4. Find the moment of inertia and the square of the radius of gyration in each of the following cases:

(Unless otherwise specified, the density in each case is uniform. The mass per unit of length, surface, or volume is denoted by m, and the total mass by M.)

(1) A thin straight rod of length l, about an axis perpendicular to its length: (a) through one end point, (b) through its middle point.

(2) A fine circular wire of radius a, about a diameter.

(3) A rectangle whose sides are $2a$, $2b$: (a) about the side $2b$, (b) about a line bisecting the rectangle and parallel to the side $2b$.

(4) A circular disc of radius a: (a) about a diameter, (b) about an axis through a point on the circumference, perpendicular to the plane of the disc, (c) about a tangent.

(5) An ellipse whose semi-axes are a and b: (a) about the major axis, (b) about the minor axis, (c) about the line through the centre at right angles to the plane of the ellipse.

(6) A semicircular area of radius a, about the diameter, the density varying as the distance from the diameter.

(7) A semicircular area of radius a, about an axis through its centre of mass, perpendicular to its plane.

(8) A rectangular parallelopiped, sides $2a$, $2b$, $2c$, about an edge $2c$.

(9) A right circular cone (height $= h$, radius of base $= b$), about its axis.

(10) A thin spherical shell of radius a, about a diameter.

(11) A sphere of radius a, about a tangent line.

(12) A right circular cylinder (length $= l$, radius $= R$): (a) about its axis, (b) about a diameter of one end.

(13) A circular arc of radius a and angle 2α: (a) about the middle radius, (b) about an axis through the centre of mass, perpendicular to the plane of the arc, (c) about an axis through the middle point of the arc, perpendicular to the plane of the arc [Lamb's *Calculus*, Exs., XXXIX.].

[**Answers:** (1) (a) $\frac{1}{3}ml^3$, $\frac{1}{3}l^2$; (b) $\frac{1}{12}ml^3$, $\frac{1}{12}l^2$. (2) $\frac{1}{2}Ma^2$, $\frac{1}{2}a^2$. (3) (a) $k^2 = \frac{4}{3}a^2$; (b) $k^2 = \frac{1}{3}a^2$. (4) (a) $k^2 = \frac{1}{4}a^2$; (b) $k^2 = \frac{5}{4}a^2$; (c) $k^2 = \frac{3}{2}a^2$. (5) (a) $\frac{1}{4}Mb^2$; (b) $\frac{1}{4}Ma^2$; (c) $\frac{1}{4}M(a^2+b^2)$. (6) $\frac{2}{5}Ma^2$, $\frac{2}{5}a^2$. (7) $k^2 = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)a^2$. (8) $k^2 = \frac{1}{5}(a^2+b^2)$. (9) $\frac{1}{10}m\pi b^4h$, $\frac{3}{10}b^2$. (10) $k^2 = \frac{2}{5}a^2$. (11) $k^2 = \frac{7}{5}a^2$. (12) (a) $I = \frac{1}{2}MR^2$; (b) $I = M(\frac{1}{4}R^2 + \frac{1}{3}l^2)$. (13) ($a$) $k^2 = \frac{1}{2}a^2\left(1 - \frac{\sin 2\alpha}{2\alpha}\right)$; ($b$) $k^2 = a^2\left(1 - \frac{\sin^2\alpha}{\alpha^2}\right)$; ($c$) $k^2 = 2a^2\left(1 - \frac{\sin\alpha}{\alpha}\right)$.]

QUESTIONS AND EXERCISES FOR PRACTICE AND REVIEW.

A large number of examples are contained in several works on calculus, in particular in those of Todhunter, Williamson, Lamb, Gibson, F. G. Taylor, and Echols. Special mention may also be made of Byerly's *Problems in Differential Calculus* (Ginn & Co.). Exercises of a practical and technical character, which are concerned with mechanics, electricity, physics, and chemistry, will be found in Perry, *Calculus for Engineers* (E. Arnold); Young and Linebarger, *Elements of the Differential and Integral Calculus* (D. Appleton & Co.); Mellor, *Higher Mathematics for Students of Chemistry and Physics* (Longmans, Green & Co.). Many of the following examples have been taken from the examination papers of various colleges and universities.

CHAPTERS II., III., IV.

1. Explain what is meant by a continuous function.

2. Explain what is meant by a discontinuous function. Give examples.

3. (1) Given that $f(x)=x^2+2$ and $F(x)=4+\sqrt{x}$, calculate $f\{F(x)\}$ and $F\{f(x)\}$. (2) If $f(x)=\dfrac{x-1}{x+1}$, show that $\dfrac{f(x)-f(y)}{1+f(x)f(y)}=\dfrac{x-y}{1+xy}$. (3) If $y=f(x)=\dfrac{2+3x}{4-7x}$ and $z=f(y)$, calculate z as a function of x. (4) If $y=\phi(x)=\dfrac{2x-1}{3x-2}$, show that $x=\phi(y)$, and show that $x=\phi^2(x)$, in which $\phi^2(x)$ is used to denote $\phi\{\phi(x)\}$, not $\{\phi(x)\}^2$. (5) If $f(x)=\dfrac{x+1}{x-1}$, show that $f^2(x)=x$, $f^3(x)=f(x)$, $f^4(x)=x$. (6) If $y=f(x)=\dfrac{ax+b}{cx-a}$, show that $x=f(y)$. (7) If $f(x,y)=ax^2+bxy+cy^2$, write $f(y,x)$, $f(x,x)$, and $f(y,y)$.

4. Define the differential coefficient of a function of x with regard to x. State what is the interpretation of the differential coefficient being positive or negative.

5. Give a geometrical interpretation of $\frac{dy}{dx}$ when x and y are connected by the relation $f(x, y) = 0$ or $y = \phi(x)$.

6. Show that the derivative of a function with respect to the variable measures the rate of increase of the function as compared with the rate of increase of the variable.

7. Find geometrically the differential coefficients of $\cos x$ and $\sin x$.

8. Deduce from first principles the first derivatives of x^n, $\sin x$, $\tan x$, $\tan^{-1} x$, $\log_a x$, a^x, $a^{\log x}$, $\log \sin \frac{x}{a}$.

9. Find the derivatives of $\frac{u}{v}$ and uv, with respect to x, where u and v are functions of x.

10. Investigate a method of finding the derivative with respect to x of a function of the form $\{f(x)\}^{\phi(x)}$, and apply it to differentiate $x^{\sqrt{1+x^2}}$.

11. Differentiate $\frac{x^{2m}}{(1+x^2)^n}$, $\frac{\log(\cos x)}{x}$, $e^{ax}\cos^r mx$, $xe^{\cos x}$, $\log\frac{b+a\cos x}{a+b\cos x}$, $e^{\tan^{-1}x}$, $\tan^{-1}e^x$, $x^m e^{ax}\sin^n x$, $\log\left(\frac{2x\sin\log x}{x^2-1}\right)$.

12. Show that (1) $D\sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}} = D\cos^{-1}\sqrt{\frac{\alpha-x}{\alpha-\beta}}$; (2) $D\sin^{-1}\frac{ax+b}{a+bx} + D\sin^{-1}\frac{\sqrt{(a^2-b^2)(1-x^2)}}{a+bx} = 0$.

13. If $x^2y^3 + \cos x - \sin x\tan y - \sin y = 0$, show that

$$\frac{dy}{dx} = \frac{(-2xy^3 + \sin x)\cos^2 y + \cos x \sin y \cos y}{3x^2y^2\cos^2 y - \sin x - \cos^3 y}.$$

14. Differentiate: (1) $\frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}$; (2) $\tan^{-1}\frac{\sqrt{b^2-a^2}\sin x}{a+b\cos x}$; (3) $\cos^{-1}\frac{b+a\cos x}{a+b\cos x}$; (4) $\sin^{-1}\frac{b+a\sin x}{a+b\sin x}$; (5) $\tan^{-1}\frac{\sqrt{a^2-b^2}\sin x}{b+a\cos x}$; (6) $\sqrt{m\sin^2 x + n\cos^2 x}$; (7) $(2a^{\frac{1}{2}} + x^{\frac{1}{2}})\sqrt{a^{\frac{1}{2}} + x^{\frac{1}{2}}}$; (8) $\frac{(\sin nx)^m}{(\cos mx)^n}$; (9) $(\cos x)^{\sin x}$; (10) $\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$.

[Answers to Ex. 14: (1) $\frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$; (2) $\frac{\sqrt{b^2-a^2}}{b+a\cos x}$; (3) $\frac{\sqrt{a^2-b^2}}{a+b\cos x}$; (4) $\frac{\sqrt{a^2-b^2}}{a+b\sin x}$; (5) $\frac{\sqrt{a^2-b^2}}{a+b\cos x}$; (6) $\frac{1}{2}(m-n)\frac{\sin 2x}{\sqrt{m\sin^2 x + n\cos^2 x}}$; (7) $\frac{4\sqrt{a}+3\sqrt{x}}{4\sqrt{x}\sqrt{a^{\frac{1}{2}}+x^{\frac{1}{2}}}}$; (8) $\frac{mn(\sin nx)^{m-1}\cos(mx-nx)}{(\cos mx)^{n+1}}$; (9) $(\cos x)^{\sin x - 1}(\cos^2 x\log\cos x - \sin^2 x)$; (10) $\frac{x}{\sqrt{1-x^4}}$.]

CHAPTER V.

1. If the equation of a plane curve be $y = \phi(x)$, find the equations of the tangent and the normal at any point, and find the lengths of the tangent, normal, subtangent, and subnormal.

2. Deduce the equation of the tangent at the point (x, y) on the curve $y = f(x)$, when the curve is given by the equations $x = \phi(t)$, $y = \psi(t)$.

Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches $y = be^{-\frac{x}{a}}$ at the point where the latter crosses the y-axis.

3. Find an equation for the normal at any point on the curve whose equation is $f(x, y) = 0$.

4. At what angle do the hyperbolas $x^2 - y^2 = a^2$ and $xy = b$ intersect? Draw sets of these curves, assigning various values to a and b.

5. Find the angle of intersection between the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$.

6. Find an expression for the angle between the tangent at any point of a curve and the radius vector to that point. Show that in the cardioid $r = a(1 + \cos\theta)$ this angle is $\frac{\pi}{2} + \frac{\theta}{2}$.

7. Determine the lengths of the tangent, normal, subtangent, and subnormal, respectively, at any point of each of the following curves: (1) the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$; (2) the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$; (3) the parabola $y^2 = 9\,x$. [*Ans.* (1) $\frac{1}{ax}\sqrt{(a^2 - x^2)(a^4 - e^2x^2)}$, $\frac{b}{a^2}\sqrt{a^4 - e^2x^2}$, $\frac{x^2 - a^2}{x}$, $\frac{b^2x}{a^2}$; (2) $\frac{y^2}{\sqrt{y^2 - a^2}}$, $\frac{y^2}{a}$, $\frac{ay}{\sqrt{y^2 - a^2}}$, $\frac{y}{a}\sqrt{y^2 - a^2}$; (3) 10, $7\frac{1}{2}$, 8, $4\frac{1}{2}$.]

8. Show that all the points of the curve $y^2 = 4\,a\left(x + a\sin\frac{x}{a}\right)$ at which the tangent is parallel to the axis of x lie on a certain parabola.

9. (1) In the curve $r = a\sin^3\frac{\theta}{3}$, show that $\phi = 4\,\psi$. (2) In the lemniscate $r^2 = a^2\sin 2\,\theta$, show that $\psi = 2\,\theta$, $\phi = 3\,\theta$, subtangent $= a\tan^2\theta\sqrt{\sin 2\,\theta}$.

10. Solve the following equations: (i) $4\,x^3 + 48\,x^2 + 165\,x + 175 = 0$; (ii) $9\,x^4 + 6\,x^3 - 92\,x^2 + 104\,x - 32 = 0$; (iii) $16\,x^5 + 104\,x^4 + 73\,x^3 - 277\,x^2 - 161\,x + 245 = 0$.

11. Show that the condition that $ax^3 + 3\,bx^2 + 3\,cx + d = 0$ may have two roots equal is $(bc - ad)^2 = 4\,(ac - b^2)(bd - c^2)$.

12. Prove, geometrically or otherwise, that provided $f(x)$ satisfies a certain condition which is to be stated

$$f(x+h)-f(x)=hf'(x+\theta h),$$

where θ is a proper fraction. Show that it is possible that in this relation θ may have more values than one.

13. If A is the area between the graph of $f(x)$, the x-axis, a fixed ordinate, and the variable ordinate $f(x)$, show that $\dfrac{dA}{dx}=f(x)$.

CHAPTER VI.

1. Find the nth derivative of the product of two functions of x in terms of the derivatives of the separate functions.

2. Find the fourth derivative of $x^5\cos^3 x$ and the nth derivatives of (i) $x^3\cos ax$; (ii) $x^4\cos^4 x$; (iii) $\tan^{-1}\dfrac{1}{x}$; (iv) $\sin^3 x\cos^2 x$; (v) $\dfrac{x^3}{x^2-1}$; (vi) $e^{ax}\sin bx$.

3. Show that

(i) $D^m\left(\dfrac{a}{x^n}\right)=(-1)^m\dfrac{n(n+1)\cdots(n+m-1)a}{x^{n+m}}$; (ii) $D^n(x^{n-1}\log x)=\dfrac{(n-1)!}{x}$;

(iii) $D^n\left(\dfrac{1-x}{1+x}\right)=\dfrac{2(-1)^n n!}{(1+x)^{n+1}}$; (iv) $D^3(e^{\sin x})=-e^{\sin x}\cos x\sin x(\sin x+3)$.

4. If $x=a(1-\cos t)$, $y=a(nt+\sin t)$, then $\dfrac{d^2y}{dx^2}=-\dfrac{n\cos t+1}{a\sin^3 t}$.

5. Derive the following: (i) If $e^y+xy-e=0$, $D_x^2 y=y\cdot\dfrac{(2-y)e^y+2x}{(e^y+x)^3}$. (ii) If $x^4+y^4+4a^2xy=0$, $(y^3+a^2x)^3\dfrac{d^2y}{dx^2}=2a^2xy(x^2y^2+3a^4)$. (iii) If $ax^2+2hxy+by^2+2gx+2fy+c=0$, $\dfrac{d^2y}{dx^2}=\dfrac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^2}$.

6. Prove the following: (i) If $y=\sin(m\tan^{-1}x)$, $(1+x^2)^2\dfrac{d^2y}{dx^2}+2x(1+x^2)\dfrac{dy}{dx}+m^2y=0$. (ii) If $y=(x+\sqrt{x^2-1})^n$, $(x^2-1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-n^2y=0$. (iii) If $y^2=\sec 2x$, $y+\dfrac{d^2y}{dx^2}=3y^5$. (iv) If $y=(1+x^2)^{\frac{m}{2}}\sin(m\tan^{-1}x)$, $(1+x^2)\dfrac{d^2y}{dx^2}-2(m-1)x\dfrac{dy}{dx}+m(m-1)y=0$.

7. If $ae^y+be^{-y}+ce^x-e^{-x}=0$, determine a relation connecting the first, second, and third derivatives of y.

CHAPTER VII.

1. Write a note on the turning values of functions of one variable.

2. Assuming $f(x)$ and its derivatives to be continuous functions, investigate the conditions that $f(a)$ should be a maximum or a minimum value of $f(x)$.

3. Show how you would proceed to find the maximum and minimum values of a single variable, and to discriminate between them.

4. If $f(x)$ have a maximum or minimum value when $x = a$, and $f'(x)$ be continuous at $x = a$, prove that $f'(x)$ must vanish when $x = a$. Show by means of a diagram that the converse is not necessarily true. Examine the case in which $f(x)$ has a maximum or minimum value when $x = a$, and $f'(x)$ is discontinuous when $x = a$.

5. If $x^3 + 3x^2y + 4y^3 = 1$, show that $\sqrt[3]{\frac{1}{4}}$ is the maximum and that $\frac{1}{2}$ is the minimum value of y, where x can have all possible values.

6. $ABCD$ is a rectangular ploughed field. A person wishes to go from A to C in the shortest possible time. He may walk across the field, or take the path along ABC; but his rate of walking on the path is double his rate of walking on the field. Show that he should make through the field for a point on BC distant $b - \dfrac{a}{\sqrt{3}}$ from C, a and b being the length of AB and BC respectively.

7. Prove that the greatest distance of the tangent to the cardioid $r = a(1 + \cos\theta)$ from the middle point of its axis is $a\sqrt{2}$.

8. AB is a fixed diameter of a circle of radius a and PQ is a chord perpendicular to AB; find the maximum value of the difference between the two triangles APQ, BPQ for different positions of the chord PQ.

9. Show that the point on the curve $4ay = x^2$, which is nearest the point $(a, 2a)$, is the point $(2a, a)$.

10. Show that the minimum value at which a normal chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ recuts the curve is $\tan^{-1}\dfrac{ab}{a^2 - b^2}$.

11. Prove that the greatest value of the area of the triangle subtended at the centre of a circle by a chord, is half the square on the radius of the circle.

12. A slip noose in a rope is thrown around a square post and the rope is drawn tight by a person standing directly before the vertical middle line of one side of the post. Show that the rope leaves the post at the angle 30°.

13. Show that the maximum and minimum values of integral algebraic functions occur alternately.

14. (i) Show that the points of inflexion on a cubical parabola $y^2 = (x - a)^2(x - b)$ lie on a line $3x + a = 4b$. (ii) Show that the curve $y(x^2 + a^2) = a^2(a - x)$ has three points of inflexion on a straight line. (iii) Show that the curve $x^3 - axy + b^3 = 0$ has a minimum ordinate at $x = \dfrac{b}{\sqrt[3]{2}}$, and a point of inflexion at $(-b, 0)$.

15. Find where the following curves have maximum or minimum ordinates and points of inflexion respectively: (i) $y = x^4 - 4x^3 - 2x^2 + 12x + 4$; (ii) $y = xe^x$; (iii) $y = xe^{-x}$; (iv) $y = xe^{-x^2}$. [*Ans.* (i) $x = -1, 1, 3, 1 \pm \frac{2}{3}\sqrt{3}$; (ii) $x = -2$; (iii) $x = 1, x = 2$; (iv) $x = \pm \frac{1}{\sqrt{2}}$, $x = 0$, $x = \pm \sqrt{\frac{3}{2}}$.]

16. Find the inflexional tangent of the curve $y = x - x^2 + x^3$. [*Ans.* $27y = 18x + 1$.]

17. Show that: (i) The cone of maximum volume for a given slant side has its semi-vertical angle $= \tan^{-1}\sqrt{2}$; (ii) The cone of maximum volume for a given total surface has its semi-vertical angle $= \sin^{-1}\frac{1}{3}$.

18. Show the march of each of the following functions: (i) $\sin^2 x \cos x$; (ii) $\sin 2x - x$; (iii) $x(a + x)^2(a - x)^3$.

19. Examine the following functions for maxima and minima:
(i) $\dfrac{x(x^2 - 1)}{x^4 - x^2 + 1}$; (ii) $\dfrac{x^2 + 2x + 11}{x^2 + 4x + 10}$; (iii) $\dfrac{1 - x + x^2}{1 + x - x^2}$; (iv) $\dfrac{1 + x + x^2}{1 - x + x^2}$;
(v) $x\sqrt{ax - x^2}$; (vi) $(x - 1)^4(x + 2)^3$; (vii) $(1 + x)^2 \div (x - x^2)$;
(viii) $\sec x - x$; (ix) $\sin x(1 + \cos x)$; (x) $a \sin x + b \cos x$; (xi) x^x;
(xii) $\dfrac{x}{\log x}$. [*Ans.* (i) Two max., each $= \frac{1}{2}$; two min., each $= -\frac{1}{2}$; (ii) max. $= 2$, min. $= \frac{5}{6}$; (iii) min. $= \frac{3}{5}$; (iv) max. $= 3$, min. $= \frac{1}{3}$; (v) min. $= \frac{3\sqrt{3}}{16}a^2$; (vi) min. $= 0$, max. $= 12^4 \cdot 9^3 \div 7^7$; (vii) max. $= 0$, min. $= 8$; (viii) $\sin x = \frac{\sqrt{5} - 1}{2}$; (ix) max. $= 1.299$; (x) max. $= \sqrt{a^2 + b^2}$, min. $= -\sqrt{a^2 + b^2}$; (xi) min. for $x = \frac{1}{e}$; (xii) min. $= e$.]

CHAPTERS VIII., IX.

1. What is meant by partial differentiation?

2. State precisely the restrictions as to the function $f(x, y)$ so that the theorem $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$ may hold, and prove the theorem.

Show that if $f(x, y) = xy\dfrac{x^2 - y^2}{x^2 + y^2}$, the theorem does not hold for $x = 0$, $y = 0$, and explain why.

3. Explain the meaning of a partial derivative. In what sense may we logically speak of the partial derivative of c with respect to a, when c is a function of a and b, and a and b are both functions of x?

4. Prove Euler's theorem for a homogeneous function ϕ of x, y, z:

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = n\phi.$$

5. If u be a homogeneous function of the nth degree in any number of variables $x, y, z, \cdots$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} + \cdots = nu$.

6. Verify that $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$ in the case of each of the following functions: $\sin(x^2 y)$, $\cos\left(\dfrac{2xy}{x^2 - y^2}\right)$, $\log\left(\dfrac{x^2 + y^2}{xy}\right)$, $\phi\left(\dfrac{x}{y}\right)$.

7. Verify the following: (i) If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$. (ii) If $u = (4ab - c^2)^{-\frac{1}{2}}$, $\dfrac{\partial^2 u}{\partial c^2} = \dfrac{\partial^2 u}{\partial a\, \partial b}$. (iii) If $z = x^2\tan^{-1}\dfrac{y}{x} - y^2\tan^{-1}\dfrac{x}{y}$, $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$. (iv) If $y = f(y + ax) + \phi(y - ax)$, in general $\dfrac{\partial^2 u}{\partial x^2} = a^2\dfrac{\partial^2 u}{\partial y^2}$. (v) If $u = \log\dfrac{x-y}{x+y} + 2\tan^{-1}\dfrac{x}{y}$, $du = \dfrac{4x^2}{x^4 - y^4}(y\,dx - x\,dy)$. (vi) If $u = \tan^{-1}\dfrac{y}{x}$, or if $u = \dfrac{xz}{x^2 + y^2}$, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$. (vii) If $u = \sin(yz + zx + xy)$, $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial z} + \dfrac{1}{1 - u^2}\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}\dfrac{\partial u}{\partial z} + 2(x + y + z)u = 0$. (viii) If $u = \sqrt{x^3 + y^3}$, $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\, \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{3}{4}u$.

8. Verify the following: (i) If $\left(3a\dfrac{dy}{dx} + 2\right)\left(\dfrac{d^2y}{dx^2}\right)^2 = \left(a\dfrac{dy}{dx} + 1\right)\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}$, $\left(\dfrac{d^2x}{dy^2}\right)^2 = \left(\dfrac{dx}{dy} + a\right)\dfrac{d^3x}{dy^3}$. (ii) If $(1 + y^2)\left(\dfrac{d^3y}{dx^3} - 2y\right) + \left(\dfrac{dy}{dx}\right)^3 = 2(1 + y)\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}$ and $y = z^2 + 2z$, $(z + 1)\dfrac{d^3z}{dx^3} = \dfrac{dz}{dx}\dfrac{d^2z}{dx^2} + z^2 + 2z$. (iii) If $\dfrac{d^2y}{dx^2} + \dfrac{2x}{1 + x^2}\dfrac{dy}{dx} + \dfrac{y}{(1 + x^2)^2} = 0$ and $x = \tan z$, $\dfrac{d^2y}{dz^2} + y = 0$. (iv) If $(a + bx)^2\dfrac{d^2y}{dx^2} + A(a + bx)\dfrac{dy}{dx} + By = F(x)$ and $a + bx = e^t$, $b^2\dfrac{d^2y}{dt^2} + b(A - b)\dfrac{dy}{dt} + By = F\left(\dfrac{e^t - a}{b}\right)$. (v) If $\dfrac{d^2y}{d\theta^2} - \sec\theta\,\mathrm{cosec}\,\theta\dfrac{dy}{d\theta} + yn^2\tan^2\theta = 0$ and $x = \log\sec\theta$, $\dfrac{d^2y}{dx^2} + n^2 y = 0$.

CHAPTERS X.-XIV.

1. Explain and illustrate the meaning of *integration*.

2. If $f(x)$ be finite and continuous for all values of x between a and b, prove that $\lim_{n=\infty} h\{f(a) + f(a + h) + f(a + 2h) + \cdots + f(a + \overline{n-1}\,h)\}$ is $\phi(b) - \phi(a)$, where $h = \dfrac{b - a}{n}$ and $\dfrac{d}{dx}\phi(x) = f(x)$.

3. Explain fully how it is that the area included between a curve, the axis of x, and two ordinates corresponding to the values x_0 and x_1 of x is represented by the definite integral $\int_{x_0}^{x_1} y\,dx$.

4. Give an outline of the reasoning by which it is shown that the area bounded by the two curves $y = \phi(x)$ and $y = \psi(x)$, and the two ordinates $x = a$ and $x = b$, is $\int_a^b \{\phi(x) - \psi(x)\} dx$.

5. Prove Simpson's or Poncelet's rule for measuring a rectangular field, one of whose sides is replaced by a curved line.

The graph of $y = x^2$ is traced on a diagram. If O be the point (0, 0) on it, P the point (10, 100), and PM the ordinate from P, find the area of OMP cut off between OM, MP, and the curve, by taking all the ordinates corresponding to integral values of the abscissas, and applying the rule you adopt. Tell exactly by how much your calculation is wrong.

6. Show how to find the volume of the surface generated by the revolution of a given curve about an axis in its plane.

7. Find the area cut off between the parabola $y = x^2$ and the circle $x^2 + y^2 = 2$.

8. Trace the curve whose equation is $a^4y^2 = x^4(a^2 - x^2)$, and find the whole area enclosed by it.

9. Show that the area included between the curve $y^2(2a - x) = x^3$ and its asymptote is $3\pi a^2$.

10. Determine the amount of area cut off from the circle whose equation is $x^2 + y^2 = 5$ by a branch of the hyperbola whose equation is $xy = 2$.

11. Trace the curve $ay + 2x(x - a) = 0$. Find the area of the closed portion contained between the curve and the axis of x. If this portion revolves round the axis of x, find the volume generated.

12. A curved quadrilateral figure is formed by the three parabolas $y^2 - 9ax + 81a^2 = 0$, $y^2 - 4ax + 16a^2 = 0$, $y^2 - ax + a^2 = 0$, the other boundary being the axis of x. Find the area of the quadrilateral.

13. Show that the volume of the solid generated by revolving about the x-axis, an arc of a parabola extending from the vertex to any point on the curve, is one-half the volume of the circumscribing cylinder.

14. Determine the curve for any point of which the subtangent is twice the abscissa and which passes through the point (8, 4).

15. Write the equation including all curves that have a constant subnormal. Determine the curve which has a constant subnormal and which passes through the points $(0, h)$, (b, k), and find what is the length of its constant subnormal. $\left[Ans.\ by^2 = (k^2 - h^2)x + bh^2;\ \frac{k^2 - h^2}{2b}.\right]$

16. In what curve is the slope at any point inversely proportional to the square of the length of the abscissa? Determine the curve which has this property and passes through (2, 5), (3, 1).

17. State and derive the rule known as "integration by parts." Apply it to find $\int x^n \log x \, dx$.

18. Show that if the integral of $f(x)$ is known, the integral of $f^{-1}(x)$, the function inverse to $f(x)$, can be found.

19. Show how to integrate $I = \dfrac{f(x)}{\phi(x)}$, where $f(x)$ and $\phi(x)$ are rational integral functions of x, and give some of the standard types for the integrals on which the value of I may be made to depend. Show how to integrate the fraction when the equation $\phi(x) = 0$ has repeated imaginary roots.

20. Show that if $f(u, v)$ is a rational function of u and v, $f\left(x, \sqrt[n]{\dfrac{ax+b}{cx+d}}\right)dx$ can be rationalised by means of the substitution $\dfrac{ax+b}{cx+d} = z^n$.

21. What is meant by *a formula of reduction* for an integral?

Investigate formulas of reduction for the following: (i) $\int \sin^m \theta \, d\theta$ in which m is an integer; (ii) $\int \sin^m \theta \cos^n \theta \, d\theta$; (iii) $\int \dfrac{x^m}{\sqrt{a^2 + x^2}} dx$; (iv) $\int x^n \sin x \, dx$.

22. Explain how it is that $\int_0^\pi \cos^{2n+1} \theta \, d\theta = 0$.

23. Evaluate $\int \dfrac{dx}{(x-p)\sqrt{ax^2 + 2bx + c}}$ by means of the substitution $y(x-p) = \sqrt{ax^2 + 2bx + c}$.

24. Evaluate the following integrals, and verify the results by differentiation: $\int \dfrac{e^{a \tan^{-1} x} dx}{(1+x^2)^{\frac{3}{2}}}$, $\int_0^a \sin^{-1}\sqrt{\dfrac{x}{a+x}}\, dx$, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{d\theta}{\sin\theta \cos^3\theta}$, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{\sqrt{\sin\theta}}{\cos^{\frac{5}{2}}\theta} d\theta$,

$\int \dfrac{d\theta}{a^2 \cos^2\theta + b^2 \sin 2\theta}$, $\int \dfrac{x^7 dx}{x^{12} - 1}$, $\int \dfrac{dx}{x(3 + 4x^5)^3}$, $\int \dfrac{dx}{3 \sin x + \sin 2x}$,

$\int x^{\frac{1}{3}}(a+x)^{\frac{2}{3}} dx$, $\int \dfrac{2x+1}{x^2 - 4x + 3} dx$, $\int x^2 \tan^{-1} x \, dx$, $\int e^{2x} \sin^2 x \, dx$,

$\int \dfrac{(\tan^{-1} x)^2}{x^3} dx$, $\int x^3 e^x dx$, $\int x^5 \log(x^3 + a^3) dx$, $\int \dfrac{(\log x)^2 dx}{x^{\frac{5}{2}}}$, $\int_0^1 \dfrac{\log x \, dx}{1+x}$,

$\int \dfrac{dx}{x\sqrt{-x^2 + 5x - 6}}$, $\int \dfrac{(x+1)dx}{\sqrt{x^2 + x + 1}}$.

CHAPTERS XV., XVI.

1. Find an expression for the area bounded by a curve given in polar coördinates and two straight lines drawn from the pole.

2. Show how to find the length of the arc of a plane curve whose equation is given (i) in rectangular Cartesian coördinates, (ii) in oblique Cartesian coördinates, (iii) in polar coördinates.

3. Investigate a formula for finding the superficial area of a surface of revolution about the axis of x.

4. Trace the curve $r^2 = a^2 \cos 3\theta$, and find the area of one of its loops.

5. Show that in the logarithmic spiral, $r = a^\theta$, the length of any arc is proportional to the difference between the vectors of its extremities.

6. Find the area of the curve $r\sqrt{a^2 + b^2} = (a^2 + b^2)\cos\theta + a^2$.

7. Find the surface of a spherical cup of height h, the radius of the sphere being R.

8. Find the average value of $\sin x \sin(\alpha - x)$ between the values 0 and α of the variable x.

9. Find the volume bounded by the surface $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1$ and the coördinate planes.

10. The axis of a cone is the diameter of a sphere through its vertex; find, in terms of its vertical angle, the volume included between the sphere and the cone, and examine for what angle it is greatest.

11. Determine the areas of each of the following figures: (i) The segment cut off from the parabola $y^2 = 4ax$ by the line $2x - 3y + 4a = 0$. (ii) The curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$. (iii) The evolute of the ellipse $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$. (iv) The figure bounded by the ellipse $16x^2 + 25y^2 = 400$, the lines $x = 2$, $x = 4$, and $2y + x = 8$. (v) The curve $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$. (vi) The oval $y = x^2 + \sqrt{(x-1)(2-x)}$. (vii) The loops of the curve $a^2y^2 = x^2(a^2 - x^2)$. (viii) The segment of the circle $x^2 + y^2 = 25$ cut off by the line $x + y = 7$. (ix) The area common to the ellipses $b^2x^2 + a^2y^2 = a^2b^2$, $a^2x^2 + b^2y^2 = a^2b^2$. [*Ans.* (i) $\frac{1}{3}a^2$. (ii) $\frac{3}{8}\pi ab$. (iii) $\frac{3}{8}\pi\frac{(a^2 - b^2)^2}{ab}$. (v) $\frac{\pi(a^2 + b^2)}{2}$. (vi) $\frac{\pi}{4}$. (vii) Each $\frac{2}{3}a^2$. (viii) $\frac{25}{2}\sin^{-1}\frac{7}{25} - \frac{7}{2}$. (ix) $4ab\tan^{-1}\frac{b}{a}$.]

12. Find the volume and the area of the surface generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line. [Area $= \frac{32}{5}\pi a^2$.]

13. Show that the volume enclosed by two right circular cylinders of equal radius a whose axes intersect at right angles is $\frac{16}{3}a^3$, and the surface of one intercepted by the other is $8a^2$.

14. Show that the volume included between the surfaces generated by the revolution of a hyperbola and its asymptotes about the transverse axis and two planes cutting this axis at right angles is the same, no matter where the sections are made, provided that the distance between the planes is kept constant.

15. The parabola $y^2 = 6x$ intersects the circle $x^2 + y^2 = 16$. Show that if the larger area intercepted between the curves revolves about the x-axis, the volume generated is 60π cubic units; and show that if the smaller area intercepted revolves about the y-axis the volume generated is $\frac{224}{5}\sqrt{3}\,\pi$ cubic units.

16. An arc of a circle of radius a revolves about its chord. Show that if the length of the chord is $2a\alpha$, volume of the solid $= 2\pi a^3(\sin\alpha - \frac{1}{3}\sin^3\alpha - \alpha\cos\alpha)$, surface of the solid $= 4\pi a^2(\sin\alpha - \alpha\cos\alpha)$.

17. Find the area of the segment cut off from the semi-cubical parabola $27ay^2 = 4(x - 2a)^3$ by the line $x = 5a$. Also find the volume and the area of the surface generated by the revolution of this segment about the x-axis.

$$\left[\textit{Ans.}\quad \tfrac{24}{5}a^2,\ \pi a^2\left\{\frac{7\sqrt{2}}{2} + \tfrac{3}{2}\log(\sqrt{2}+1)\right\}.\right]$$

18. A number n is divided at random into two parts. Show that the mean value of the sum of their squares is $\frac{2}{3}n^2$.

19. Show that the mean of the squares on the diameters of an ellipse, that are drawn at points on the curve whose eccentric angles differ successively by equal amounts, is equal to one-half the sum of the squares on the major and minor axes.

20. Prove that the mean distance of the points of a spherical surface of radius a from a point P at a distance c from the centre is $c + \dfrac{a^2}{3c}$ or $a + \dfrac{c^2}{3a}$, according as P is external or internal.

CHAPTER XVII.

1. Define curvature of a curve. Find an expression for the radius of curvature of a curve whose equation is in the form $y = f(x)$.

2. Show that the curvature at any point of the curve given by $x = \phi(t)$, $y = \psi(t)$ is $\dfrac{\phi'\psi'' - \psi'\phi''}{(\phi'^2 + \psi'^2)^{\frac{3}{2}}}$, where accents denote differentiations with respect to t.

3. For any curve $f(r, \theta) = 0$ show that radius of curvature $= \dfrac{r}{\sin\psi\cdot\left(1 + \dfrac{d\psi}{d\theta}\right)}$, in which $\psi = \tan^{-1}\dfrac{r\,d\theta}{dr}$.

4. Find the coördinates of the point on the parabola $x^2 = 4ay$ for which the radius of curvature is equal to the latus rectum.

5. Show that at a point of undulation the tangent has contact of at least the third order.

6. Show that the circle $(4x - 3a)^2 + (4y - 3a)^2 = 8a^2$ and the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ have contact of the third order at the point $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$. Find the order of contact of the curves $y = x^3$ and $y = 3x^2 - 3x + 1$.

7. Show that the circles of curvature of the parabola $y^2 = 4\,ax$ for the ends of the latus rectum have for their equations $x^2 + y^2 - 10\,ax \pm 4\,ay - 3\,a^2 = 0$, and that they cut the curve again in the points $(9\,a, \mp 6\,a)$.

8. Find the radius of curvature of each of the following curves: (i) The cardioid $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta$. (ii) $y = 2\,x + 3\,x^2 - 2\,xy + y^2$ at $(0, 0)$. (iii) $xy^2 = a^2(a + x)$ at $(-a, 0)$. (iv) The tractrix $x = a \log \cot \frac{\theta}{2} - a \cos \theta$, $y = a \sin \theta$. (v) $y = x - \sin x$ at the origin, and where $x = \frac{\pi}{2}$. (vi) The exponential curve $y = ae^{\frac{x}{c}}$. (vii) $r^m = a^m \cos m\theta$. (viii) $r = a \sin n\theta$ at $(0, 0)$. (ix) $r^3 = a^3 \cos 3\,\theta$. [*Ans.* (i) $\frac{2}{3}\sqrt{ar}$. (ii) $\frac{5}{6}\sqrt{5}$. (iii) $\frac{1}{2}\,a$. (iv) $-a \cot \theta$. (v) $0, 2\sqrt{2}$. (vi) $\frac{(c^2 + y^2)^{\frac{3}{2}}}{cy}$. (vii) $\frac{a^m}{(m + 1)r^{m-1}}$. (viii) $\frac{1}{2}\,na$. (ix) $\frac{a^3}{4\,r^2}$.]

CHAPTER XVIII.

1. Define an asymptote to a curve. Derive a method of finding the asymptotes of an algebraic curve whose equation in Cartesian coördinates is of the nth degree.

2. Show that the asymptotes of the cubic $x^2y - xy^2 + y^2 + xy + x - y = 0$ cut the curve again in three points which lie on the line $x + y = 0$.

3. Find the asymptotes of the curve $xy^2 - x^3 + 2\,x^2 + 3\,y + x - 1 = 0$. Show that the points at a finite distance from the origin in which the asymptotes cut the curve lie on the line $3\,y + 2\,x - 1 = 0$.

4. Draw the curve $x^2y = x^3 - a^3$. Show that it has an asymptote which crosses the x-axis at an angle $\tan^{-1} 3$.

5. Find the asymptotes of the following curves: (i) $xy^2 - x^2y = a^2(x + y) + b^3$. (ii) $1 + y = e^{\frac{1}{x}}$. (iii) $x^3 - xy^2 + ay^2 - a^2y = 0$. (iv) $(x^2 + y^2)(y^2 - 4\,x^2) + 4\,y^2(x - 1) + x^2(4\,x + 3) = 0$. (v) $(x - 2\,a)y^2 = x^3 - a^3$. (vi) $x^3 + 3\,y^3 = a^2(y - x)$. (vii) $x^3 + 2\,x^2y + xy^2 - x^2 - xy + 2 = 0$. (viii) $r \sin 2\,\theta = a \cos 3\,\theta$. (ix) $y^3 = x^2(2\,a - x)$.

6. Find the asymptotes of the curve $x^3y - xy^3 + 6\,a^2xy + a^2y - 16\,a^2x = 0$. Show that the origin is a point of inflexion.

7. Define *a family of* (plane) *curves*, and *the variable parameter* of the family. Define the envelope of a family of curves. Define an ultimate intersection of a family of curves. Define the locus of the ultimate intersections of a family of curves. Illustrate the definitions by concrete examples and diagrams, and furnish any explanations you may think necessary.

8. Show that in general the locus of ultimate intersections of the family touches each member of the family. Show that this locus is, in general, the envelope of the family. Explain the necessity of the qualifying phrase "in general."

9. Explain the method of finding the envelopes of the curves $f(x, y, t)=0$, where t is a variable parameter.

10. Write a note on "singular points of curves," explaining what they are, giving illustrations, and showing how to find them.

11. Ellipses of equal area are described with their axes along fixed straight lines. Show that the envelope consists of two equilateral hyperbolas.

12. Prove that the circles which pass through the origin and have their centres on the equilateral hyperbola $x^2 - y^2 = a^2$ envelop the lemniscate $(x^2 + y^2)^2 = 4a^2(x^2 - y^2)$.

13. P is a point on a parabola of which A is the vertex. Find the equation of the curve touched by all circles described on AP as diameter.

14. A circle passes through the origin, and its centre lies on the parabola $y^2 = 4ax$. Show that the envelope of all such circles is a cissoid.

15. A straight line moves so that the product of the perpendiculars on it from two fixed points $(\pm c, 0)$ is constant $(= k^2)$. Show that its envelope is the ellipse $\frac{x^2}{k^2 + c^2} + \frac{y^2}{k^2} = 1$, or the hyperbola $\frac{x^2}{c^2 - k^2} - \frac{y^2}{k^2} = 1$.

16. Find the envelope of circles passing through the centre of an ellipse $a^2y^2 + b^2x^2 = a^2b^2$ and having centres on the circumference of the ellipse. [*Ans.* $(x^2 + y^2)^2 = 4(a^2x^2 + b^2y^2)$.]

17. Ellipses are described having their axes coincident in direction with those of a given ellipse, and lengths of axes proportional to the coördinates of a variable point on the given ellipse. Show that the ellipses all touch four straight lines.

18. Find the equation of the envelope of the line $x \sin \alpha + y \cos \alpha = a \sin \alpha \cos \alpha$.

19. From a fixed point on the circumference of a circle chords are drawn, and on these as diameters circles are drawn. Show that the envelope of the series of circles is a cardioid.

20. If a cannon is fired at an elevation θ, and the projectile has an initial velocity equal to that attained by a body in falling h feet, the equation of the parabolic path, referred to horizontal and vertical axes through the point of projection, is $y = x \tan \theta - \frac{x^2}{4h} \sec^2 \theta$. Find the envelope of the paths for different elevations.

CHAPTERS XIX., XX.

1. A function $f(x)$ is defined by an infinite series $f(x) = \sum_{n=1}^{n=\infty} \phi_n(x)$; state and prove a sufficient condition that the equation $\frac{d}{dx} f(x) = \sum_{n=1}^{n=\infty} \frac{d}{dx} \phi_n(x)$ may be true.

2. Write a note on the conditions under which (1) the integral, (2) the differential coefficient of an infinite series, may be obtained by integrating or differentiating the series term by term.

3. Prove that if $f(x)$ be a continuous function of x, then

$$f(x+h) = f(x) + hf'(x+\theta h),$$

where $0 < \theta < 1$.

Show clearly how this proposition may be applied to prove Taylor's theorem, and specify the circumstances in which the theorem as you state it is true.

4. Prove Taylor's theorem for the expansion of $f(x+h)$ in ascending powers of h, carefully specifying the conditions which $f(x)$ must satisfy. Find an expression for the remainder after n terms of the series have been written down.

5. State Maclaurin's theorem, and give the conditions under which it is applicable to the expansion of functions. Derive the theorem.

6. Expand in series of ascending powers of x the functions: (i) $\cos mx$. (ii) $\tan^{-1}(a+x)$. (iii) $\sin(m \sin^{-1} x)$. (iv) $(1+y)^x$, where $y < 1$. (v) $e^{mx} + e^{-mx}$. (vi) $e^{\sqrt{x+h}}$, 4 terms.

7. Expand the following functions in powers of x: (i) $e^{\sin x}$. (ii) $\tan^{-1} x$. (iii) $\cot^{-1} x$. [*Ans.* (i) $1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{15}x^5 + \cdots$. (ii) For values of x from $x = -1$ to $x = 1$, $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$; for $|x| > 1$, $\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots$. (iii) For $|x| < 1$, $\frac{\pi}{2} - x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \cdots$; for $|x| > 1$, $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots$.]

8. Calculate the values of the following:

(i) $\int_0^x x^{\frac{1}{3}}\sqrt{1-x^2}\,dx$. (ii) $\int_0^x x\cot x\,dx$. (iii) $\int_{-1}^{+1} e^{x^2}\,dx$. (iv) $\int_0^x e^x \sin x\,dx$. (v) $\int_0^x \frac{\sin x}{x}\,dx$. [*Ans.* (i) $\frac{3}{4}x^{\frac{4}{3}}(1 - \frac{1}{5}x^2 - \frac{1}{32}x^4 - \frac{1}{88}x^6 + \cdots)$. (ii) $x - \frac{x^3}{9} - \frac{x^5}{225} - \frac{2x^7}{6615}\cdots$. (iii) $2\left(1 + \frac{1}{3} + \frac{1}{1\cdot2\cdot5} + \frac{1}{1\cdot2\cdot3\cdot7} + \frac{1}{1\cdot2\cdot3\cdot4\cdot9} + \cdots\right)$. (iv) $\frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{2x^4}{4!} - \frac{2^2x^6}{6!} - \frac{2^3x^7}{7!} - \frac{2^3x^8}{8!} + \cdots$. (v) $x - \frac{x^3}{3\cdot3!} + \frac{x^5}{5\cdot5!} - \cdots$.]

CHAPTER XXI.

1. Solve the following equations:

(1) $x^2y\,dx - (x^3+y^3)dy = 0$. (2) $3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$.
(3) $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$. (4) $x\,Dy - y = x\sqrt{x^2+y^2}$.
(5) $(x^2+y^2)(x\,dx + y\,dy) = a^2(x\,dy - y\,dx)$. (6) $(x^2+1)Dy + 2xy = 4x^2$.
(7) $6(x+1)Dy = y - y^4$. (8) $p^3 - 4xyp + 8y^2 = 0$, in which $p = D_xy$.
(9) $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$. (10) $\frac{dy}{dx} + \frac{1-2x}{x^2}y = 1$. (11) $y = x^2 - \frac{1}{2}p^2$. (12) $x + 2py = p^2x$.

(13) $D_x^3y + 2D_x^2y + D_xy = 0.$ (14) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 0.$

(15) $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 0.$ (16) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0.$ (17) $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$ (18) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4\left(\frac{dy}{dx}\right)^3 = 0.$ (19) $2y^2\frac{d^2y}{dx^2} = 1.$

(20) $y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx} - 2y\right) = 0.$ (21) $2xD^3y\,D^2y + a^2 = (D^2y)^2.$

[*Solutions:* (1) $3y^3\log y = x^3 + c.$ (2) $\tan y = c(1 - e^x)^3.$ (3) $x^3 - 6x^2y - 6xy^2 + y^3 = c.$ (4) $2y = x(ce^x - ce^{-x}).$ (5) $x^2 + y^2 = 2a^2\tan^{-1}\frac{y}{x} + c.$ (6) $3(x^2+1)y = 4x^3 + c.$ (7) $\sqrt{x+1}(1 - y^3) = cy^3.$ (8) $y = c(x-c)^2.$ (9) $2y^{-5} = cx^5 + 5x^3.$ (10) $y = x^2(1 + ce^x).$ (11) $(x^2+y)^2(x^2-2y) + 2x(x^2-3y)c = c^2.$ (12) $1 + 2cy = c^2x^2.$ (13) $y = c_1 + e^{-x}(c_2 + c_3x).$ (14) $y = e^x(c_1 + c_2\cos x + c_3\sin x).$ (15) $y = c_1 + c_2x + e^x(c_3 + c_4x).$ (16) $xy = c_1\log x - \log(x-1) + c_2.$ (17) $y = x(c_1 + c_2\log x) + c_3x^{-1}.$ (18) $\sin(c_1 - 2\sqrt{2}\,y) = c_2e^{-2x}.$ (19) $x = \frac{1}{c}\sqrt{cy^2 - y} + \frac{1}{2c\sqrt{c}}\text{hycos}^{-1}(2cy - 1) + c_1.$ (20) $2x = \log(y^2 + c_1) + c_2.$ (21) $15c_1^2y = 4(c_1x + a^2)^{\frac{5}{2}} + c_2x + c_3.$]

2. Find the singular solutions of:
(1) $x^2p^2 - 3xyp + 2y^2 + x^3 = 0.$ (2) $xp^2 - 2yp + ax = 0.$ (3) Solve equation (2).
[*Solutions:* (1) $x^2(y^2 - 4x^3) = 0.$ (2) $y^2 = ax^2.$ (3) $2y = cx^2 + \frac{a}{c}.$]

MISCELLANEOUS.

1. How far does the symbol $\frac{d}{dx}$ obey the fundamental laws of algebra?

2. Prove that if D denote $\frac{d}{dx}$, and $f(D)$ be any rational algebraic function of D, then $f(D)uv = uf(D)v + Duf'(D)v + \frac{D^2u}{2!}f''(D)v + \cdots.$

3. If ϕ denote any function of x, prove that $\frac{d^n(x\phi)}{dx^n} = n\frac{d^{n-1}\phi}{dx^{n-1}} + x\frac{d^n\phi}{dx^n}.$ By this theorem or otherwise find the value of $D^5(x\sin mx)$.

4. If $x = e^\theta$, prove that $\frac{d}{d\theta}\left(\frac{d}{d\theta} - 1\right)\left(\frac{d}{d\theta} - 2\right)\cdots\left(\frac{d}{d\theta} - n + 1\right)u = x^n\frac{d^nu}{dx^n},$ where u is any function of x. Prove also that $\left(\frac{d}{dx}x\frac{d}{dx}\right)^n u = \left(\frac{d}{dx}\right)^n x^n \left(\frac{d}{dx}\right)^n u.$

5. If $\phi(x)$ is a function involving positive integral powers of x, prove the symbolic equation $\phi\left[\frac{d}{dx}\left(e^{ax}\cdot u\right)\right] = e^{ax}\phi\left(a + \frac{d}{dx}\right)u.$

6. Show how to find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when x and y are connected by the equation $f(x, y) = 0.$

7. If $u = f(x, y)$ and if $x = \phi(t)$, $y = \psi(t)$, state and prove the rule for obtaining the total derivative of u with respect to t.

If $x = r\cos\theta$, $y = r\sin\theta$, transform $(x^2 - y^2)\dfrac{\partial^2 u}{\partial x\,\partial y} + xy\left(\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2}\right)$ into an expression in which r and θ are the independent variables.

8. Calculate the nth derivative of $(\sin^{-1}x)^2$. Show by the use of Maclaurin's theorem that $(\sin^{-1}x)^2 = 2\left(\dfrac{x^2}{2} + \dfrac{2}{3 \cdot 4}\dfrac{x^4}{} + \dfrac{2 \cdot 4}{3 \cdot 5 \cdot 6}x^6 + \cdots\right)$.

9. The curves $u = 0$, $u' = 0$ intersect at (x, y) at an angle α. Show that

$$\tan\alpha = \frac{\dfrac{\partial u}{\partial x}\dfrac{\partial u'}{\partial y} - \dfrac{\partial u'}{\partial x}\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial u'}{\partial x}\dfrac{\partial u}{\partial y}}.$$

Show that the curves $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a'^2} + \dfrac{y^2}{b'^2} = 1$ intersect at right angles if $a^2 - b^2 = a'^2 - b'^2$.

10. Show that the total surface of a cylinder inscribed in a right circular cone cannot have a maximum value if the semi-angle of the cone exceeds $\tan^{-1}\frac{1}{2}$, *i.e.* 26° 34′.

11. Through a diameter of the base of a right circular cone are drawn two planes cutting the cone in parabolas. Show that the volume included between these planes and the vertex is $\dfrac{4}{3\pi}$ of the volume of the cone.

12. Calculate the area common to the cardioid $r = a(1 - \cos\theta)$ and the circle of radius $\frac{3}{2}a$ whose centre is at the pole.

13. Find the area and the perimeter of the smaller quadrilateral bounded by the circles $x^2 + y^2 = 25$, $x^2 + y^2 = 144$, and the parabolas, $y^2 = 8x$, $y^2 + 12(x + 2) = 0$.

14. Given the cardioid $r = 4(1 - \cos\theta)$ and the circle of radius 6 whose centre is at the cusp, find the length of the circular arc inside the cardioid and the lengths of the arcs of the cardioid which are respectively outside the circle and inside the circle.

15. If a curve be defined by the equations $\dfrac{x}{\phi(t)} = \dfrac{y}{\psi(t)} = \dfrac{1}{f(t)}$, find an expression for the radius of curvature at a point whose parameter is t.

16. Expand (by any method) $x^3\operatorname{cosec}^3 x$ in a series of powers of x as far as the term in x^4. At what place of decimals may error come in by stopping at this term, when x is less than a right angle?

17. Trace the curve $x^4 + y^4 = a^2xy$, and find the points at which the tangent is parallel to an axis of coördinates. Find the area of the loop.

18. Trace the curve $x = a\sin 2\theta(1 + \cos 2\theta)$, $y = a\cos 2\theta(1 - \cos 2\theta)$. ($a$) Prove that θ is the angle which the tangent makes with the axis of x, and obtain the equation of the tangent to the curve. (b) Find the length of the radius of curvature in terms of θ.

19. Find $\frac{dy}{dx}$ under each of the following conditions: (i) $x^3 = e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$. (ii) $y = e^{x^x} \tan^{-1} x$. (iii) $e^x + x = e^y + y$. (iv) $y = \frac{1}{x + \sqrt{1 - x^2}}$. (v) $\sin(xy) - e^{xy} - x^2y = 0$.

20. Four circles $x^2 + y^2 = 2\,ax$, $x^2 + y^2 = 2\,ay$, $x^2 + y^2 = 2\,bx$, $x^2 + y^2 = 2\,by$, form by their intersections in the first quadrant a quadrilateral; prove that the area of this is $(a^2 + b^2) \cot^{-1} \frac{2\,ab}{a^2 - b^2} - (a - b)^2$.

21. Prove that the area of a sector of an ellipse of semi-axes a and b between the major axis and a radius vector from the focus is $\frac{ab}{2}(\phi - e \sin \phi)$, where ϕ is the eccentric angle of the point to which the radius vector is drawn.

22. Trace the curve $xy^3 = a^4$; and find whether the area between it, a given ordinate, and the coördinate axes is finite.

Show also that if the tangent at P meet the axis of x in T, then $MT = 3\,OM$, where M is the foot of the ordinate at P, and O is the origin.

23. If u be a homogeneous function of n dimensions in x and y, show that:
(i) $x^2 \frac{\partial^2 u}{\partial x^2} + 2\,xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$. (ii) $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x\, \partial y} = (n-1) \frac{\partial u}{\partial x}$.
(iii) $x \frac{\partial^2 u}{\partial x\, \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n - 1) \frac{\partial u}{\partial y}$. (iv) $\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)^2 u = n^2 u$.

24. Prove the following: (i) If $u = \sin^{-1}(xyz)$, $\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} = \tan^2 u \sec u$.
(ii) If $u = \log(\tan x + \tan y + \tan z)$, $\sin 2\,x \frac{\partial u}{\partial x} + \sin 2\,y \frac{\partial u}{\partial y} + \sin 2\,z \frac{\partial u}{\partial z} = 2$.
(iii) If $u = \log(x^3 + y^3 + z^3 - 3\,xyz)$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$. (iv) If $u = \tan^2 x \tan^2 y \tan^2 z$, $du = 4\,u\left(\frac{dx}{\sin 2\,x} + \frac{dy}{\sin 2\,y} + \frac{dz}{\sin 2\,z}\right)$.

25. If b be the radius of the middle section of a cask, a the radius of either end, and h its length, show that the volume of the cask is $\frac{1}{15}\pi(3\,a^2 + 4\,ab + 8\,b^2)h$, assuming that the generating curve is an arc of a parabola.

26. OM is the abscissa, MP the ordinate of a point $P(x_1, y_1)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, ($x_1$, y_1, both being positive). If A is the vertex nearest P, show that area $AMP = \frac{1}{2}\,x_1y_1 - \frac{1}{2}\,ab \log\left(\frac{x_1}{a} + \frac{y_1}{b}\right)$, and area sector $OAP = \frac{1}{2}\,ab \log\left(\frac{x_1}{a} + \frac{y_1}{b}\right)$.

27. Show that the mean of the squares on the diameters of an ellipse that are drawn at equal angular intervals is equal to the rectangle contained by the major and minor axes.

28. Find the mean square of the distance of a point within a square from the centre of the square.

29. Through a diameter of one end of a right circular cylinder of altitude h and radius a two planes are passed touching the other end on opposite sides. Show that the volume included between the planes is $(\pi - \frac{4}{3})a^2h$.

30. Show that the integration of the expression $f(x, y)dx\,dy$ may be performed in any order, provided the limits of x and y are independent of each other.

31. Evaluate $\int\int\int x^\alpha y^\beta z^\gamma\,dx\,dy\,dz$ taken throughout the space bounded by the coördinate planes and the plane $x + y + z = 1$.

32. Prove geometrically or otherwise that $x\,dy - y\,dx = r^2\,d\theta$, and show that the area of a closed curve is represented by $\frac{1}{2}\int(x\,dy - y\,dx)$.

33. The equation to a curve being written in terms of the polar coördinates r and θ, p being the perpendicular from the pole to the tangent and $u = \frac{1}{r}$, show that, $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$.

34. If a is a first approximation to a root of the equation $f(x) = 0$, determine graphically or otherwise the conditions under which $a - \frac{f(a)}{f'(a)}$ is a valid second approximation.

35. If $f(x)$ be a finite and continuous function of x between $x = a$ and $x = b$, show that a value x_1 of x, lying between a and b, may be found such that $f'(x_1) = \{f(b) - f(a)\} \div (b - a)$.

If the function be $x^c + cx$, find the point in question when $a = \alpha$ and $b = 2\alpha$, and thence show that in this case x_1 is such that $\frac{a - x_1}{b - x_1}$ is constant for all values of α.

36. Find the radius of curvature of the curves: (i) limaçon $r = a\cos\theta + b$, where $r = \frac{b}{2}$; (ii) $ay^2 = (x-a)(x-b)^2$ at $(a, 0)$. Trace the curves. [*Ans.* (i) $\frac{2a^3}{4a^2 - b^2}$; (ii) $\frac{(a-b)^2}{2a}$.]

37. (1) Trace the curve $r = a + b\cos\theta$, $a > b > 0$; find its area. (2) Find the area of the loop of $y^2 = (x-1)(x-3)^2$. (3) Find the area between the x-axis and one arch of the harmonic curve $y = b\sin\frac{x}{a}$. [*Ans.* $\frac{1}{2}(2a^2 + b^2)\pi$, $\frac{32\sqrt{2}}{15}$, $2ab$.]

38. Trace the curve $9y^2 = (x+7)(x+4)^2$. Find the area and the length of the loop, and the volume and area of the surface generated by the revolution of the loop about the x-axis. [*Ans.* $\frac{8}{5}\sqrt{3}$, $4\sqrt{3}$, $\frac{9}{4}\pi$, 3π.]

39. Find the limiting values of: (i) $\log \dfrac{\pi^2 \sin\theta}{(\pi^2-\theta^2)\theta}$, when $\theta=\pi$; (ii) $\left(\dfrac{\log x}{x}\right)^{\frac{1}{x}}$, when $x=\infty$; (iii) $\dfrac{x^n-x}{1-x+\log x}$, when $x=1$; (iv) $\dfrac{1}{2x^2}-\dfrac{\pi}{2x\tan\pi x}$, when $x=0$; (v) $\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$, when $x=0$; (vi) $\dfrac{a^x-b^x}{x}$, when $x=0$; (vii) $\dfrac{e^x-e^a}{x^2-a^2}$, when $x=a$.

40. Find the mass of an elliptic plate of semi-axes a and b, the density varying directly as the distance from the centre and also as the distances from the principal axes.

41. From a fixed point A on the circumference of a circle of radius a, the perpendicular AY is let fall on the tangent at P. Prove that the greatest area APY can have is $\dfrac{3\sqrt{3}}{8}a^2$.

42. A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned up so as to form an open rectangular box. Show that the box has a maximum volume when its depth is $\frac{1}{6}(a+b-\sqrt{a^2-ab+b^2})$, a and b being the sides of the original rectangle.

43. Two ships are sailing uniformly with velocities u, v, along straight lines inclined at an angle θ: show that if a, b, be their distances at one time from the point of intersection of the courses, the least distance of the ships is equal to

$$\frac{(av-bu)\sin\theta}{(u^2+v^2-2uv\cos\theta)^{\frac{1}{2}}}.$$

44. A right circular conical vessel 12 inches deep and 6 inches in diameter at the top is filled with water: calculate the diameter of a spherical ball which, on being put into the vessel, will expel the most water.

45. A statue a feet high is on a pedestal whose top is b feet above the level of the observer's eyes. How far from the pedestal should the observer stand in order to get the best view of the statue? [*Ans.* $\sqrt{b(a+b)}$ feet.]

46. The lower corner of a leaf, whose width is a, is folded over so as just to reach the inner edge of the page: find the width of the part folded over when (1) the length of the crease is a minimum, (2) when the area of the triangle folded over is a minimum. [*Ans.* (1) $\frac{3}{4}a$; (2) $\frac{2}{3}a$.]

47. (1) Show that the cylinder of greatest volume for a given surface has its height equal to the diameter of the base, and its volume equal to .8165 of that of the sphere of equal surface.

(2) Show that the cylinder of least surface for a given volume has its height equal to its diameter, and its surface equal to 1.1447 of that of the sphere of equal volume.

48. Trace the graph of $y = \dfrac{\sin 2x - \sin x}{\cos x}$. Find the angles at which it crosses the x-axis, and show that its finite maximum distance from the x-axis is $(2^{\frac{2}{3}} - 1)^{\frac{3}{2}}$.

49. An ellipse, whose centre is at the origin and whose principal axes coincide with the axes of x and y, touches the straight line $qx+py=pq$; find the semi-axes when the area of the ellipse is a maximum, and also the coördinates of its point of contact with the given line.

50. Find the volume of the greatest parcel of square cross-section which can be sent by parcel post, the Post-office regulations being that the length plus girth must not exceed 6 feet, while the length must not exceed 3 feet 6 inches.

INTEGRALS.

FOR EXERCISE AND REVIEW.

The following list of integrals provides useful exercises in formal differentiation and integration. It will also afford some assistance in the solution of practical problems as a table of reference. Those who have to make considerable use of the calculus will find it a great advantage to have at hand Peirce's *Short Table of Integrals* * (Ginn & Co.).

GENERAL FORMULAS OF INTEGRATION.

Formulas A, B, C, pages 173, 174; formula for integration by parts, page 177.

FUNDAMENTAL ELEMENTARY INTEGRALS.

Formulas I.-XXVI., pages 172, 173, 180, 181. (These should be memorised.)

REDUCTION FORMULAS FOR $\int x^{\pm m}(a + bx^n)^{\pm p}\,dx$.

[Here X denotes $(a + bx^n)$.]

1. $$\int x^m X^p\,dx = \frac{x^{m-n+1}X^{p+1}}{b(np+m+1)} - \frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}X^p\,dx.$$

2. $$\int x^m X^p\,dx = \frac{x^{m+1}X^{p+1}}{a(m+1)} - \frac{b(m+n+np+1)}{a(m+1)}\int x^{m+n}X^p\,dx.$$

3. $$\int x^m X^p\,dx = \frac{x^{m+1}X^p}{m+np+1} + \frac{anp}{m+np+1}\int x^m X^{p-1}\,dx.$$

4. $$\int x^m X^p\,dx = -\frac{x^{m+1}X^{p+1}}{an(p+1)} + \frac{m+n+np+1}{an(p+1)}\int x^m X^{p+1}\,dx.$$

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* There are two editions, the briefer edition of 32 pages and the revised edition of 134 pages.

5. $\displaystyle\int x^m X^p\,dx = \frac{x^{m-n+1}X^{p+1}}{bn(p+1)} - \frac{m-n+1}{bn(p+1)}\int x^{m-n}X^{p+1}\,dx.$

6. $\displaystyle\int x^m X^p\,dx = \frac{x^{m+1}X^p}{m+1} - \frac{bnp}{m+1}\int x^{m+n}X^{p-1}\,dx.$

7. $\displaystyle\int \frac{dx}{x^m X^p} = -\frac{1}{(m-1)ax^{m-1}X^{p-1}} - \frac{(m-n+np-1)b}{(m-1)a}\int\frac{dx}{x^{m-n}X^p}.$

8. $\displaystyle\int \frac{dx}{x^m X^p} = \frac{1}{an(p-1)x^{m-1}X^{p-1}} + \frac{m-n+np-1}{an(p-1)}\int\frac{dx}{x^m X^{p-1}}.$

9. $\displaystyle\int \frac{X^p\,dx}{x^m} = -\frac{X^{p+1}}{a(m-1)x^{m-1}} - \frac{b(m-n-np-1)}{a(m-1)}\int\frac{X^p\,dx}{x^{m-n}}.$

10. $\displaystyle\int \frac{X^p\,dx}{x^m} = \frac{X^p}{(np-m+1)x^{m-1}} + \frac{anp}{np-m+1}\int\frac{X^{p-1}\,dx}{x^m}.$

11. $\displaystyle\int \frac{x^m\,dx}{X^p} = \frac{x^{m-n+1}}{b(m-np+1)X^{p-1}} - \frac{a(m-n+1)}{b(m-np+1)}\int\frac{x^{m-n}\,dx}{X^p}.$

12. $\displaystyle\int \frac{x^m\,dx}{X^p} = \frac{x^{m+1}}{an(p-1)X^{p-1}} - \frac{m+n-np+1}{an(p-1)}\int\frac{x^m\,dx}{X^{p-1}}.$

13. $\displaystyle\int \frac{dx}{(a+bx^2)^n} = \frac{1}{2(n-1)a}\left[\frac{x}{(a+bx^2)^{n-1}} + (2n-3)\int\frac{dx}{(a+bx^2)^{n-1}}\right].$

Put a^2 for a, $b=1$, and compare with Ex. 3, Art. 118.

14. $\displaystyle\int \frac{x^2\,dx}{(a+bx^2)^n} = \frac{-x}{2b(n-1)(a+bx^2)^{n-1}} + \frac{1}{2b(n-1)}\int\frac{dx}{(a+bx^2)^{n-1}}.$

15. $\displaystyle\int \frac{dx}{x^2(a+bx^2)^n} = \frac{1}{a}\int\frac{dx}{x^2(a+bx^2)^{n-1}} - \frac{b}{a}\int\frac{dx}{(a+bx^2)^n}.$

EXPRESSIONS CONTAINING $\sqrt{a+bx}$.

Also see Ex. 10, page 191.

16. $\displaystyle\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}.$

17. $\displaystyle\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}.$

EXPRESSIONS CONTAINING $\sqrt{x^2 \pm a^2}$.

Also see Ex. 7, page 191.

18. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(\frac{x + \sqrt{x^2 \pm a^2}}{a}\right)$. See XXIV., XXV., page 181.

19. $\int (x^2 \pm a^2)^{\frac{n}{2}} dx = \frac{x(x^2 \pm a^2)^{\frac{n}{2}}}{n+1} \pm \frac{na^2}{n+1} \int (x^2 \pm a^2)^{\frac{n}{2}-1} dx.$

20. $\int (x^2 \pm a^2)^{\frac{1}{2}} dx = \frac{x}{2} \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2}).$

21. $\int (x^2 \pm a^2)^{\frac{3}{2}} dx = \frac{x}{8} (2\,x^2 \pm 5\,a^2) \sqrt{x^2 \pm a^2} + \frac{3\,a^4}{8} \log (x + \sqrt{x^2 \pm a^2}).$

22. $\int x^2(x^2 \pm a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2\,x^2 \pm a^2) \sqrt{x^2 \pm a^2} - \frac{a^4}{8} \log (x + \sqrt{x^2 \pm a^2}).$

23. $\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \pm \frac{x}{a^2 \sqrt{x^2 \pm a^2}}.$

24. $\int \frac{x^2\, dx}{(x^2 \pm a^2)^{\frac{1}{2}}} = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2}).$

25. $\int \frac{x^2\, dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = - \frac{x}{\sqrt{x^2 - a^2}} + \log (x + \sqrt{x^2 - a^2}).$

26. $\int \frac{dx}{x(x^2 + a^2)^{\frac{1}{2}}} = \frac{1}{a} \log \frac{x}{a + \sqrt{x^2 + a^2}}; \quad \int \frac{dx}{x(x^2 - a^2)^{\frac{1}{2}}} = \frac{1}{a} \sec^{-1} \frac{x}{a}.$

27. $\int \frac{dx}{x^2(x^2 \pm a^2)^{\frac{1}{2}}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$

28. *a.* $\int \frac{dx}{x^3(x^2 + a^2)^{\frac{1}{2}}} = - \frac{\sqrt{x^2 + a^2}}{2\,a^2 x^2} + \frac{1}{2\,a^3} \log \frac{a + \sqrt{x^2 + a^2}}{x}.$

b. $\int \frac{dx}{x^3(x^2 - a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2 - a^2}}{2\,a^2 x^2} + \frac{1}{2\,a^3} \sec^{-1} \frac{x}{a}.$

29. *a.* $\int \frac{(x^2 + a^2)^{\frac{1}{2}} dx}{x} = \sqrt{x^2 + a^2} - a \log \frac{a + \sqrt{a^2 + x^2}}{x}.$

b. $\int \frac{(x^2 - a^2)^{\frac{1}{2}} dx}{x} = \sqrt{x^2 - a^2} - a \cos^{-1} \frac{a}{x}.$

30. $\int \frac{(x^2 \pm a^2)^{\frac{1}{2}} dx}{x^2} = - \frac{\sqrt{x^2 \pm a^2}}{x} + \log (x + \sqrt{x^2 \pm a^2}).$

EXPRESSIONS CONTAINING $\sqrt{a^2 - x^2}$.

Also see Ex. 7, page 191.

31. $$\int (a^2 - x^2)^{\frac{n}{2}} dx = \frac{x(a^2 - x^2)^{\frac{n}{2}}}{n+1} + \frac{a^2 n}{n+1}\int (a^2 - x^2)^{\frac{n}{2}-1} dx.$$

32. $$\int \frac{x^m dx}{\sqrt{a^2 - x^2}} = -\frac{x^{m-1}\sqrt{a^2 - x^2}}{m} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2}dx}{\sqrt{a^2 - x^2}}.$$

33. $$\int x^m \sqrt{a^2 - x^2}\, dx = \frac{x^{m+1}\sqrt{a^2 - x^2}}{m+2} + \frac{a^2}{m+2}\int \frac{x^m dx}{\sqrt{a^2 - x^2}}.$$

34. $$\int \frac{dx}{x^m \sqrt{a^2 - x^2}} dx = -\frac{\sqrt{a^2 - x^2}}{(m-1)a^2 x^{m-1}} + \frac{m-2}{(m-1)a^2}\int \frac{dx}{x^{m-2}\sqrt{a^2 - x^2}}.$$

35. $$\int \frac{\sqrt{a^2 - x^2}}{x^m} dx = -\frac{\sqrt{a^2 - x^2}}{(m-2)x^{m-1}} - \frac{a^2}{m-2}\int \frac{dx}{x^m \sqrt{a^2 - x^2}}.$$

36. $$\int (a^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

37. $$\int (a^2 - x^2)^{\frac{3}{2}} dx = \frac{x}{8}(5\, a^2 - 2\, x^2)\sqrt{a^2 - x^2} + \frac{3\, a^4}{8}\sin^{-1}\frac{x}{a}.$$

38. $$\int x^2 (a^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{8}(2\, x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}.$$

39. $$\int \frac{x^2\, dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

40. $$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$$

41. $$\int \frac{x^2\, dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a}.$$

42. $$\int \frac{dx}{x^2 (a^2 - x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$$

43. $$\int \frac{dx}{x(a^2 - x^2)^{\frac{1}{2}}} = \frac{1}{a}\log\frac{x}{a + \sqrt{a^2 - x^2}}.$$

44. $$\int \frac{dx}{x^3 (a^2 - x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2 - x^2}}{2\, a^2 x^2} + \frac{1}{2\, a^3}\log\frac{x}{a + \sqrt{a^2 - x^2}}.$$

45. $$\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x} dx = \sqrt{a^2 - x^2} - a\log\frac{a + \sqrt{a^2 - x^2}}{x}.$$

46. $$\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} = -\sin^{-1}\frac{x}{a}.$$

EXPRESSIONS CONTAINING $\sqrt{2\,ax - x^2}$, $\sqrt{2\,ax + x^2}$.

[Here X denotes $\sqrt{2\,ax - x^2}$, and Z denotes $\sqrt{2\,ax + x^2}$.]

47. a. $\int \frac{dx}{X} = \sin^{-1}\frac{x-a}{a}$. b. $\int \frac{dx}{Z} = \log(x + a + Z)$.

48. a. $\int X\,dx = \frac{x-a}{2}X + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}$.

b. $\int Z\,dx = \frac{x+a}{2}Z - \frac{a^2}{2}\log(x + a + Z)$.

49. a. $\int x^m X\,dx = -\frac{x^{m-1}X^3}{m+2} + \frac{(2\,m+1)a}{m+2}\int x^{m-1}X\,dx$.

b. $\int x^m Z\,dx = \frac{x^{m-1}Z^3}{m+2} - \frac{(2\,m+1)a}{m+2}\int x^{m-1}Z\,dx$.

50. a. $\int \frac{dx}{x^m X} = -\frac{X}{(2\,m-1)ax^m} + \frac{m-1}{(2\,m-1)a}\int \frac{dx}{x^{m-1}X}$.

b. $\int \frac{dx}{x^m Z} = \frac{-Z}{(2\,m-1)ax^m} - \frac{m-1}{(2\,m-1)a}\int \frac{dx}{x^{m-1}Z}$.

51. a. $\int \frac{x^m\,dx}{X} = -\frac{x^{m-1}X}{m} + \frac{(2\,m-1)a}{m}\int \frac{x^{m-1}\,dx}{X}$.

b. $\int \frac{x^m\,dx}{Z} = \frac{x^{m-1}Z}{m} - \frac{(2\,m-1)a}{m}\int \frac{x^{m-1}\,dx}{Z}$.

52. a. $\int \frac{X}{x^m}\,dx = -\frac{X^3}{(2\,m-3)ax^m} + \frac{m-3}{(2\,m-3)a}\int \frac{X}{x^{m-1}}\,dx$.

b. $\int \frac{Z}{x^m}\,dx = -\frac{Z^3}{(2\,m-3)ax^m} - \frac{m-3}{(2\,m-3)a}\int \frac{Z}{x^{m-1}}\,dx$.

53. a. $\int xX\,dx = -\frac{3\,a^2 + ax - 2\,x^2}{6}X + \frac{a^3}{2}\sin^{-1}\frac{x-a}{a}$.

b. $\int xZ\,dx = -\frac{3\,a^2 - ax - 2\,x^2}{6}Z + \frac{a^3}{2}\log(x + a + Z)$.

54. a. $\int \frac{dx}{xX} = -\frac{X}{ax}$. b. $\int \frac{dx}{xZ} = -\frac{Z}{ax}$.

55. a. $\int \frac{x\,dx}{X} = -X + a\sin^{-1}\frac{x-a}{a}$. b. $\int \frac{x\,dx}{Z} = Z - a\log(x + a + Z)$.

56. a. $\int \frac{x^2\,dx}{X} = -\frac{x+3\,a}{2}X + \frac{3}{2}a^2\sin^{-1}\frac{x-a}{a}$.

b. $\int \frac{x^2\,dx}{Z} = \frac{x-3\,a}{2}Z + \frac{3}{2}a^2\log(x + a + Z)$.

57. *a.* $\int \frac{X\,dx}{x} = X + a\sin^{-1}\frac{x-a}{a}$. *b.* $\int \frac{Z\,dx}{x} = Z + a\log(x+a+Z)$.

58. *a.* $\int \frac{X}{x^2}dx = -\frac{2X}{x} - \sin^{-1}\frac{x-a}{a}$. *b.* $\int \frac{Z}{x^2}dx = -\frac{2Z}{x} + \log(x+a+Z)$.

59. *a.* $\int \frac{X}{x^3}dx = -\frac{X^3}{3ax^3}$. *b.* $\int \frac{Z}{x^3}dx = -\frac{Z^3}{3ax^3}$.

60. *a.* $\int \frac{dx}{X^3} = \frac{x-a}{a^2X}$. *b.* $\int \frac{dx}{Z^3} = -\frac{x+a}{a^2Z}$.

61. *a.* $\int \frac{x\,dx}{X^3} = \frac{x}{aX}$. *b.* $\int \frac{x\,dx}{Z^3} = \frac{x}{aZ}$.

EXPRESSIONS CONTAINING $a + bx \pm cx^2$.

62. *a.* $\int \frac{dx}{a+bx+cx^2} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}}$, for $b^2 < 4ac$

$$= \frac{1}{\sqrt{b^2-4ac}}\log\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}},\text{ for } b^2 > 4ac.$$

b. $\int \frac{dx}{a+bx-cx^2} = \frac{1}{\sqrt{b^2+4ac}}\log\frac{\sqrt{b^2+4ac}+2cx-b}{\sqrt{b^2+4ac}-2cx+b}$.

63. *a.* $\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\log(2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2})$.

b. $\int \frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}}\sin^{-1}\frac{2cx-b}{\sqrt{b^2+4ac}}$.

64. *a.* $\int \sqrt{a+bx+cx^2}\,dx = \frac{2cx+b}{4c}\sqrt{a+bx+cx^2}$

$$-\frac{b^2-4ac}{8c^{\frac{3}{2}}}\log(2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}).$$

b. $\int \sqrt{a+bx-cx^2}\,dx = \frac{2cx-b}{4c}\sqrt{a+bx-cx^2} + \frac{b^2+4ac}{8c^{\frac{3}{2}}}\sin^{-1}\frac{2cx-b}{\sqrt{b^2+4ac}}$.

65. *a.* $\int \frac{x\,dx}{\sqrt{a+bx+cx^2}} = \frac{\sqrt{a+bx+cx^2}}{c}$

$$-\frac{b}{2c^{\frac{3}{2}}}\log(2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}).$$

b. $\int \frac{x\,dx}{\sqrt{a+bx-cx^2}} = -\frac{\sqrt{a+bx-cx^2}}{c} + \frac{b}{2c^{\frac{3}{2}}}\sin^{-1}\frac{2cx-b}{\sqrt{b^2+4ac}}$.

N.B. Other algebraic integrals that are occasionally useful are given in Exs. 7–10, page 191, and in Exs. 4, 6, page 222.

EXPONENTIAL AND TRIGONOMETRIC EXPRESSIONS.

The most elementary of these are given in the integrals on pages 172, 180.

66. *a.* $\int \sin x \cos^n x\,dx = -\frac{\cos^{n+1}x}{n+1}$. *b.* $\int \sin^n x \cos x = \frac{\sin^{n+1}x}{n+1}$.

67. *a.* $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x$. *b.* $\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x$.

68. $$\int \sin^n x\,dx = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

69. $$\int \cos^n x\,dx = \frac{\cos^{n-1}x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

70. $$\int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\,\frac{\cos x}{\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2}x}.$$

71. $$\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\,\frac{\sin x}{\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}x}.$$

72. $$\int \sec^n x\,dx = \frac{\tan x \sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx. \quad \text{(Cf. 71.)}$$

73. $$\int \operatorname{cosec}^n x\,dx = -\frac{\cot x \operatorname{cosec}^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2}x\,dx. \quad \text{(Cf. 70.)}$$

74. $$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx.$$

75. $$\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx.$$

76. $$\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m-1}x \cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x \cos^n x\,dx.$$

77. $$\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1}x \cos^{n+1}x}{m+1} + \frac{m+n+2}{m+1}\int \sin^{m+2}x \cos^n x\,dx.$$

78. $$\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1}x \cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2}x\,dx.$$

79. $$\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m+1}x \cos^{n+1}x}{n+1} + \frac{m+n+2}{n+1}\int \sin^m x \cos^{n+2}x\,dx.$$

80. $\int \sin mx \sin nx \, dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}.$

81. $\int \cos mx \cos nx \, dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}.$

82. $\int \sin mx \cos nx \, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}.$

83. $$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right), \text{ when } a > b$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}, \text{ when } a < b.$$

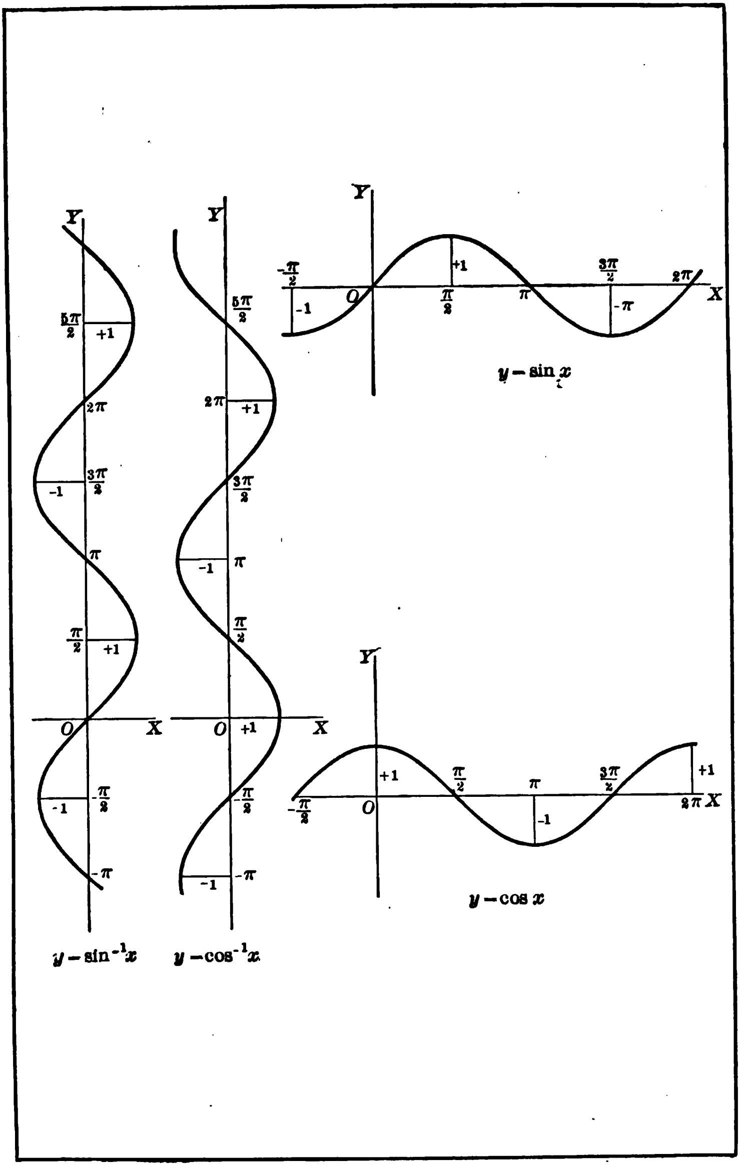
84. $$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}, \text{ when } a > b$$

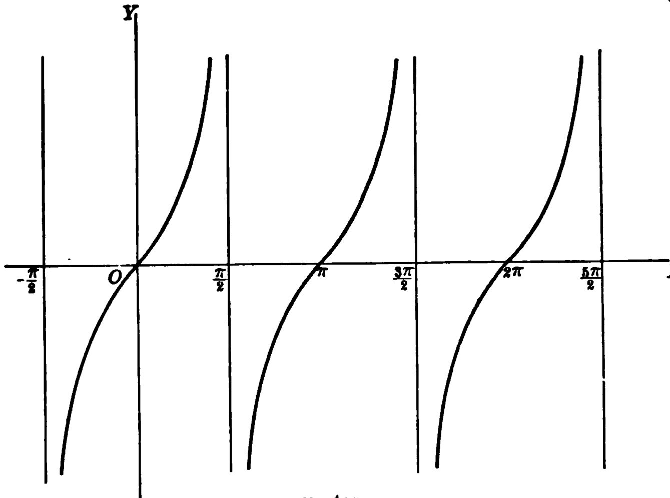
$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}, \text{ when } a < b.$$

85. $$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right).$$

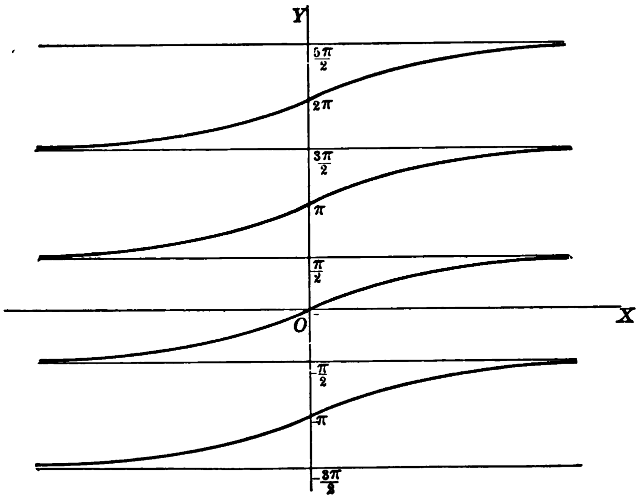
86. $\int e^{ax} \sin nx \, dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2+n^2}.$ (See Ex. 19, Art. 106.)

87. $\int e^{ax} \cos nx \, dx = \frac{e^{ax}(n\sin nx + a\cos nx)}{a^2+n^2}.$ (See Ex. 6, Art. 106.)

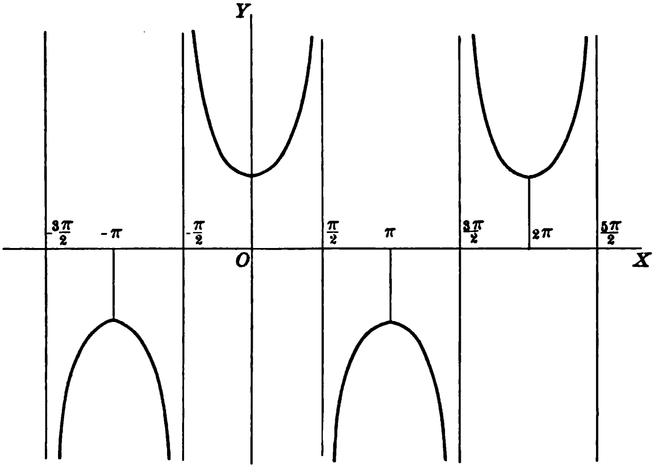

$y = \sin^{-1}x$
$y = \cos^{-1}x$
$y = \sin x$
$y = \cos x$



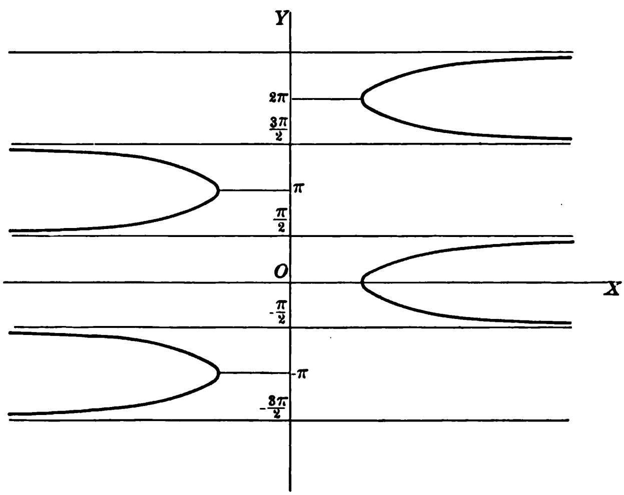
$y = \tan x$



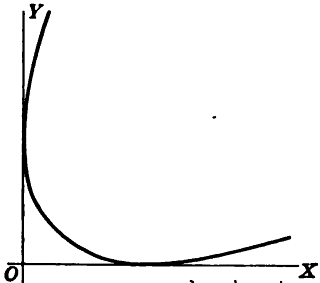
$y = \tan^{-1} x$



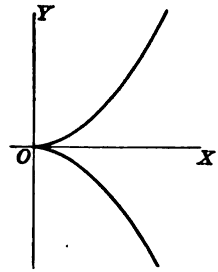
$y = \sec\ x$



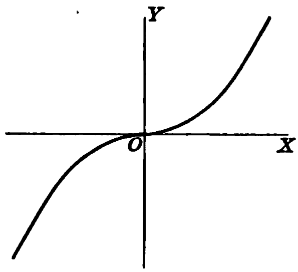
$y = \sec^{-1} x$



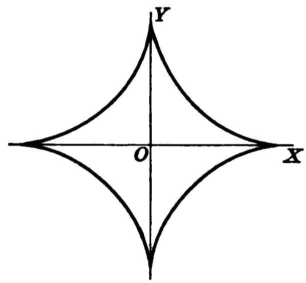
The Parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$



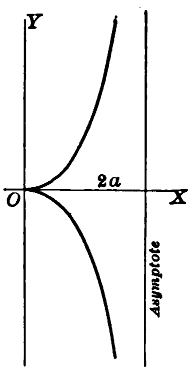
The Semi-Cubical Parabola,
$a\,y^2 = x^3$



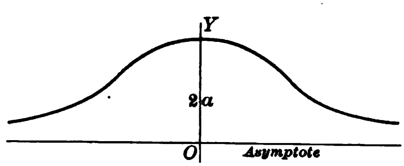
The Cubical Parabola $a^2 y = x^3$



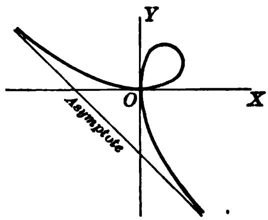
The Astroid or Four-Cusped
Hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$



The Cissoid of Diocles
$y^2 = \frac{x^3}{2a - x}$

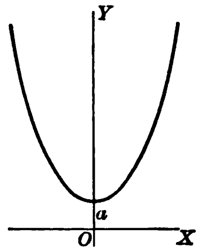


The Witch of Agnesi
$y = \frac{8a^3}{x^2 + 4a^2}$



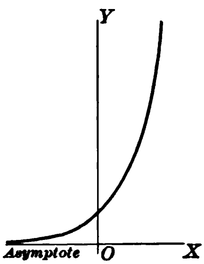
The Folium of Descartes

$x^3 + y^3 = 3\,a\,x\,y$



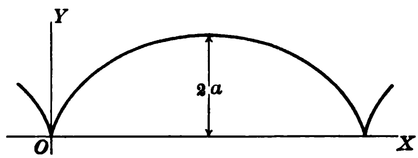
The Catenary

$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$



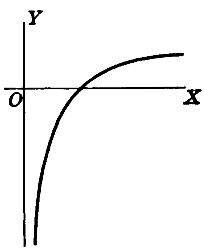
The Exponential Curve

$y = e^x$



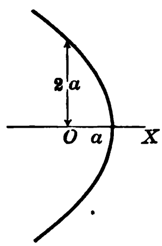
The Cycloid

$x = a\,(\theta - \sin\theta),\ y = a\,(1 - \cos\theta)$



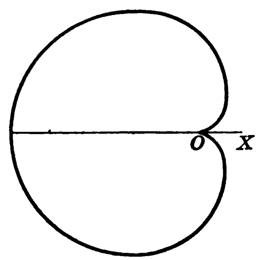
The Logarithmic Curve

$y = \log\, x$



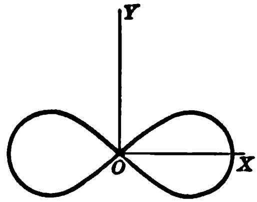
Parabola

$r = a \sec^2 \frac{\theta}{2}$

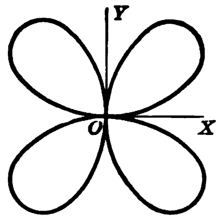


The Cardioid

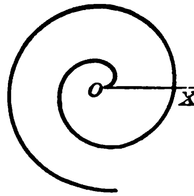
$r = a(1 - \cos\theta)$



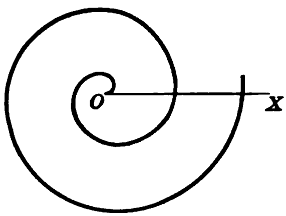
The Lemniscate, $r^2 = a^2 \cos 2\theta$,
$(x^2 + y^2)^2 = a^2 (x^2 - y^2)$



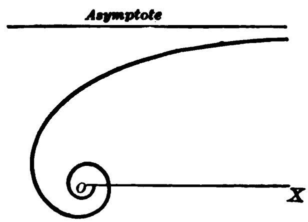
The Curve, $r = a \sin 2\theta$



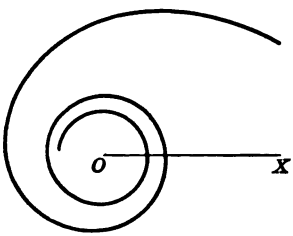
The Parabolic Spiral
$r^2 = a^2 \theta$



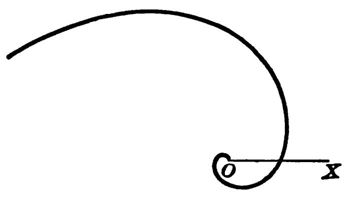
The Spiral of Archimedes, $r = a\theta$



The Hyperbolic or Reciprocal
Spiral, $r\theta = a$



The Lituus or Trumpet,
$r^2 \theta = a^2$



The Logarithmic or Equiangular
Spiral, $r = e^{a\theta}$ or $\log r = a\theta$

ANSWERS TO THE EXAMPLES.

CHAPTER I.

Art. 4. **1.** 45°, 0°, 63° 26′ 4″, 71° 33′ 54″, 75° 57′ 49″, 78° 41′ 24″, 80° 32′ 16″, 82° 52′ 30″, 104° 2′ 11″, 99° 27′ 44″, 135°, 126° 52′.2, 110° 33′.3. **2.** (.18, .033), (.29, .83), (.5, .25), (.87, .75), (5.72, 32.66), (− 1.07, 1.15), (− .35, .12), (− .18, .033), (− .09, .008). **3.** [The latter part.] (*a*) $-\frac{x}{y}$; (*b*) $2x + 1$; (*c*) $3x^2$; (*d*) $\frac{4}{y}$; (*e*) $\frac{-9x}{16y}$; (*f*) $\frac{9x}{16y}$; (*g*) $\frac{2p}{y}$; (*h*) $-\frac{b^2x}{a^2y}$; (*i*) $\frac{b^2x}{a^2y}$. **4.** ***a.*** ∞, ± .5774, ± .2582, 0, ± .4045, ± 1.8074; 90°, 30° and 150°, 14° 28′.7 and 165° 31′.3, 0°, 22° 1′.4 and 157° 58.′6, 61° 2′.7 and 118° 57′.3. ***b.*** 27, 12, 3, 0, 6.75, 18.75; 87° 52′.7, 85° 14′.2, 71° 33′.9, 0°, 81° 34′.4. 86° 56′.8. ***c.*** ∞, ± 1.4142, ± 1, ± .8165, ± .5774, ± .5; 90°, 54° 44′.1 and 125° 15′.9, 45° and 135°, 39° 14′ and 140° 46′, 30° and 150°, 26° 34′ and 153° 26′. ***d.*** 0, ± .1937, ± .4330, ∞, ± .0945, ± .3034; 0°, 10° 57′.7 and 169° 2′.3, 23° 24′.8 and 156° 35′.2, 90°, 5° 24′ and 174° 36′, 16° 52′.7 and 163° 7′.3. ***e.*** ∞, ± .8661, ± .8183, ± 1.25, ± .9139; 90°, 40° 53′.8 and 139° 6′.2, 39° 17.′6 and 140° 42.′4, 51° 20′.4 and 128° 39′.6, 42° 25′.4 and 137° 34′.6. **5.** Where $x = \pm 2.57$; where $x = \pm 2.78$.

CHAPTER II.

Art. 12. **1.** 35.2426 or 26.7574, 29.9586 or 28.0614, $3\sqrt{\sin x} + \frac{2}{\sin x} + 7\sin^2 x + 2$. **2.** 68, 28, 14, $3\sin^2 x - 5\sin x + 21$. **3.** $\frac{14 - 5x}{2 - 49x}$. **4.** $18 + 8\sqrt{x} + x$, $4 + \sqrt{x^2 + 2}$. **5.** $ay^2 + bxy + cx^2$, $(a + b + c)x^2$, $(a + b + c)y^2$.

CHAPTER III.

Art. 22. **4.** (*a*) $2x$, $2x$, $2x$; (*b*) $3x^2$, $3x^2$, $3x^2$. **5.** $4x^3$, $2x + 4$, $-\frac{1}{x^2}$, $-\frac{2}{x^2} - 3 + 4x$. **6.** $6t$, $12t^2 - 8 - \frac{3}{t^2}$. **7.** $6y^5$, $\frac{3}{2}y - 8 + \frac{7}{y^2}$.

Art. 26. **2.** $2\pi r$ times, r being the measure of the radius; 1.51 sq. in. per second; 2.83 sq. in. per second. **3.** .866 a times, a being the measure of the side; 25.98 and 51.96 sq. in. per second. **4.** $4\pi r^2$ times, r being the measure of the radius; 9.425 and 37.7 cu. in. per second. **5.** $5\frac{31}{61}$ mi. per hour.

Art. 27. **3.** $3x^2\,dx$, dx, $2\,dx$, $3\,dx$, $a\,dx$, $2x\,dx$, $14x\,dx$, etc. **4.** 1.6; 1.681. **5.** 42.2; 43.696. *Ex.* 5.03 and 9.425 sq. in. *Ex.* 1.3 and 2.6 sq. in.

CHAPTER IV.

Art. 31. $6x^2 + 14x - 10$, $2x - 17$, $-2x + 21$.

Art. 32. **4.** $(5x^4 - 8x^3 + 21x^2 + 2x - 2)\,dx$, ….

Art. 33. **1.** $\dfrac{3x^4 - 14x^3 + 6x^2}{(3x^2 - 7x + 2)^2}$, $\dfrac{16x - 21x^2 - x^4}{(x^3 + 8)^2}$, $\dfrac{-2x^2 + 44x - 96}{(2x^2 - 9x + 3)^2}$, $\dfrac{(3x^4 - 14x^3 + 6x^2)\,dx}{(3x^2 - 7x + 2)^2}$, …. **2.** ∞, $-\dfrac{17}{640}$, $\dfrac{-8}{245}$.

Art. 35. **2.** $\dfrac{4t(3t^2 - 4)}{4t + 17}$. **3.** $\dfrac{14x^3}{3x + 7}$.

Art. 37. **1.** $2u\dfrac{du}{dx}$, $12u^3\dfrac{du}{dx}$, $63u^8\dfrac{du}{dx}$, $8x^7$, $12x^3$, $84x^{11}$, $27x^2 - 34x + 10$. **3.** $240x(5x^2 - 10)^{23}$, $120x^3(3x^4 + 2)^9$, $(432x^5 + 300x^3 - 168x^2 + 448x - 50)(4x^2 + 5)^7(3x^4 - 2x + 7)^4$. **4.** $-2u^{-3}u'$, $-7u^{-8}u'$, $-11u^{-12}u'$, $-7x^{-8}$, $-15x^{-6}$, $-170x^{-11}$, $-8x(x^2 - 3)^{-5}$, $-60x^3(3x^4 + 7)^{-6}$, $15x^4 - 21x^2 + \dfrac{1}{x^2} - \dfrac{10}{x^3} + \dfrac{3}{x^4}$. **5.** $\frac{1}{2}u^{-\frac{1}{2}}Du$, $-\frac{3}{4}u^{-\frac{7}{4}}Du$, $\frac{7}{5}u^{\frac{2}{5}}Du$, $\frac{1}{2}x^{-\frac{1}{2}}$, $\frac{7}{2}x^{\frac{5}{2}}$, $\frac{5}{2}x^{\frac{3}{2}}$, $\dfrac{3x}{\sqrt{3x^2 - 5}}$, $\dfrac{4x + 7}{3}(2x^2 + 7x - 3)^{-\frac{2}{3}}$, $\dfrac{1}{\sqrt{2x + 7}}$, $-\frac{2}{5}(3x - 7)^{-\frac{8}{5}}$, $6x - \frac{7}{2}x^{-\frac{1}{2}} - x^{-\frac{3}{2}} - 2x^{-\frac{5}{3}} + \frac{4}{35}x^{-\frac{7}{5}}$. **6.** $\sqrt{2}\,u^{\sqrt{2}-1}u'$, $\sqrt{3}\,x^{\sqrt{3}-1}$, $5\sqrt{7}\,x^{\sqrt{7}-1}$, $2\sqrt{5}(2x + 5)^{\sqrt{5}-1}$, $\sqrt{3}(6x + 7)(3x^2 + 7x - 4)^{\sqrt{3}-1}$. **7.** $\dfrac{x^4}{4} + c$, and give c any three particular constant values. **8.** (In each of these expressions k is to be given *any* three particular constant values.) $\dfrac{x^6}{6} + k$, $-\dfrac{1}{x} + k$, $\dfrac{2}{3}x^{\frac{3}{2}} + k$, $\dfrac{2}{5}x^{\frac{5}{2}} + k$, $\dfrac{6}{5}x^5 + \dfrac{2}{x} - 2\sqrt{x} + k$. **12.** $6x^2 + 34x - 61$, $max^{m-1} - nbx^{-n-1}$, $\dfrac{4x}{(1 - x^2)^2}$, $\dfrac{-2a}{(a + x)^2}$, $\dfrac{x}{\sqrt{1 + x^2}}$, $-\dfrac{12}{x^5} + \dfrac{5}{3}x^{-\frac{2}{3}} - 35x^4$, $-\dfrac{1}{x^2\sqrt{1 + x^2}}$, $\dfrac{a}{(a - bx^2)^{\frac{3}{2}}}$, $\dfrac{3x^2}{(1 - x^2)^{\frac{5}{2}}}$, $\dfrac{1}{(1 - x)\sqrt{1 - x^2}}$, $mnx^{n-1}(1 + x^n)^{m-1}$, $12bx^2(a + bx^3)^3$, $x^{m-1}(1 - x)^{n-1}[m - (m + n)x]$, $\dfrac{a - 3x}{2\sqrt{a - x}}$. **14.** *a.* $\dfrac{ay - x^2}{y^2 - ax}$, $\dfrac{4x(x^2 + ay)}{a(3y^2 - 2x^2)}$, $\dfrac{9x^2y - 8x - 14xy^2 - 2y^3}{14x^2y + 6xy^2 - 3x^3 - 16y}$, $\dfrac{-(x + a)y^2}{(a + y)(b^2 - ay - 2y^2) + y(x + a)^2}$, $-\dfrac{x}{y}$, $-\dfrac{b^2x}{a^2y}$. *b.* $-\frac{3}{4}$, $\frac{3}{4}$, $\frac{3}{4}$, $-\frac{3}{4}$. **17.** $y = x^2 + k$, in which k is an arbitrary constant; $y = x^2 + 1$. **18.** 5 ft. per second. **19.** 10 mi. an hour; $8\frac{2}{3}$ ft. per second. **20.** (4, 8). **21.** 3 hr.; 60 mi. **22.** $\frac{5}{4}$ ft. per second. **23.** 36° 52′.2. **24.** 36° 52′.2.

Art. 39. **1.** $\frac{(6x+4)\log_a e}{3x^2+4x-7}$, $\frac{6x+4}{3x^2+4x-7}$, $\frac{.434(6x+4)}{3x^2+4x-7}$, $\frac{11}{16\log_e a}$, $\frac{11}{16}$, .29858. **2.** $\frac{1}{3}$, .144765. **3.** $\frac{-2}{1-x^2}$, $\frac{1}{1-x^2}$, $\frac{1}{(1-x)\sqrt{x}}$, $\frac{1}{\sqrt{x^2+a^2}}$, $\frac{1}{x\log x}$, $1+\log x$. **4.** $\log(x^2+3x+5)+c$, $\log c(x^5-7x-1)$, $\log\sqrt{kx}$, in which c and k are arbitrary constants. (*Ex.* Write each of these anti-derivatives with the arbitrary constant involved in other ways.)

6. (*a*) $\frac{-(2167+1877x+228x^2)\sqrt{x+2}}{30(4x-7)^{\frac{7}{5}}(3x+5)^{\frac{3}{5}}}$, (*b*) $\frac{6(x^2-2)}{(x+1)^2(x+2)^2}$,

(*c*) $\frac{91x^2+475x+450}{15(2x+5)^{\frac{1}{2}}(7x-5)^{\frac{2}{3}}(x+3)^{\frac{7}{5}}}$.

Art. 40. **1.** $2xe^{x^2}$, $2.303(10^x)$, $2.303(6x\cdot10^{3x^2})$, $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$. **2.** $2e^{2t}$, $2.303(2t\cdot10^{t^2})$, $2te^{t^2+3}$, $4.606(10^{2t+7})$. **3.** $e^x x^{m-1}(x+m)$, $na^{x^n}\cdot x^{n-1}\log a$, $\frac{e^x(1-x)-1}{(e^x-1)^2}$, $(1-x)e^{-x}$, $\frac{4}{(e^x+e^{-x})^2}$, $e^{x^2}\left(2-\frac{1}{x^2}\right)$. **4.** $\frac{1}{3}e^{3x}+c$, $\frac{1}{2}e^{x^2}+c$, $\frac{2}{3}e^{3x+1}+c$, c being an arbitrary constant.

Art. 41. **2.** $(3x+7)^{x^2}\left[2x\log(3x+7)+\frac{3x^2}{3x+7}\right]$, $(3x+7)^{2x}\left[\log(3x+7)^2+\frac{6x}{3x+7}\right]$, as last, $\sqrt[x]{x}\left(\frac{1-\log x}{x^2}\right)$, $x^{x^n}\cdot x^{n-1}(n\log x+1)$, $e^{e^x}\cdot e^x$, $-\frac{1}{e}\left(\frac{e}{x}\right)^{\frac{x}{e}}\log x$, $\frac{1}{x}-\log a$.

Art. 42. **1.** $\frac{d}{dx}\sin 2u=\cos 2u\cdot\frac{d}{dx}(2u)=2\cos 2u\cdot\frac{du}{dx}$, $3\cos 3u\cdot Du$, $\frac{1}{2}\cos\frac{1}{2}u\cdot u'$, $\frac{2}{3}\cos\frac{2}{3}u\frac{du}{dx}$, $\frac{17}{2}\cos\frac{17}{2}u\cdot Du$. **2.** $D\sin 2x=\cos 2x\cdot D(2x)=2\cos 2x$, $3\cos 3x$, $\frac{1}{2}\cos\frac{1}{2}x$, $6x\cos 3x^2$, $3\sin 6x$, $20x^4\cos 4x^5$, $20\sin^4 4x\cos 4x$. **3.** $5\cos 5t$, $t\cos\frac{1}{2}t^2$. **4.** $\frac{2\cos 2x\sin 3x-3\sin 2x\cos 3x}{\sin^2 3x}$, $\sin 2x+2x\cos 2x$, $2x\sin\left(x+\frac{\pi}{4}\right)+x^2\cos\left(x+\frac{\pi}{4}\right)$. **5.** 45° and 135°. **6.** Where $x=n\pi\pm\cdot 9553$, in which n is any integer. **7.** 63° 26′ and 116° 34′. **8.** Where $x=n\pi-\frac{\pi}{4}$, in which n is any integer; 54° 44′.1 and 125° 15′.9; where $x=n\pi+\frac{\pi}{4}$, n being any integer. **9.** $n\cos nx$, $nx^{n-1}\cos x^n$, $n\sin^{n-1}x\cos x$, $2x\cos(1+x^2)$, $n\cos(nx+\alpha)$, $nbx^{n-1}\cos(a+bx^n)$, $12\sin^2 4x\cos 4x$, $\frac{x\cos x-\sin x}{x^2}$, $\frac{\cos(\log x)}{x}$, $\cot x$, $e^x\cos(e^x)\cdot\log x+\frac{\sin e^x}{x}$.
10. (*a*) $\sin x+c$, $\frac{1}{3}\sin 3x+c$, $\frac{1}{2}\sin(2x+5)+c$, $\frac{1}{2}\sin(x^2-1)+c$, in which c is an arbitrary constant. (*b*) $\frac{1}{2}\sin 2x+c$, $\frac{1}{3}\sin(3x-7)+c$, $\frac{1}{3}\sin x^3+c$, in which c is any constant.

Art. 43. **3.** Where $x = n\pi$, n being an integer; where $x = (4n-1)\frac{\pi}{2}$ $\pm \cdot 485$, $2n\pi - \cdot 485$. **5** $-\frac{b}{a}\cot\theta$. **6.** $\cot\frac{\theta}{2}$; 60°. **7.** $-2\sin(2x+5)$, $-15\cos^2 5x\sin 5x$, $2x\cos x - x^2\sin x$, $\frac{2\sin x}{(1+\cos x)^2}$, $-(m\cos nx\sin mx$ $+n\cos mx\sin nx)$, $e^{\cos x}(1-x\sin x)$, $e^{ax}(a\cos mx - m\sin mx)$. **8.** $-\cos x + c$, $-2\cos\frac{1}{2}x + c$, $-\frac{1}{3}\cos(3x-2)+c$, $-\frac{1}{2}\cos(x^2+4)+c$; c being an arbitrary constant.

Art. 44. **3.** $2\sec^2 2u\cdot Du$, $3\sec^2 3u\cdot Du$, $m\sec^2 mu\cdot u'$, $2nu\sec^2 nu^2\cdot u'$, $2\sec^2 2x$, $\frac{1}{2}\sec^2\frac{1}{2}x$, $m\sec^2 mx$, $6x\sec^2 3x^2$, $12x^2\sec^2 4x^3$, $nmx^{n-1}\sec^2 mx^n$, $6\tan 3x\sec^2 3x$, $12\tan^2 4x\sec^2 4x$, $nm\tan^{n-1}mx\sec^2 mx$, $\frac{4}{3}\tan(\frac{2}{3}x+3)\sec^2$ $(\frac{2}{3}x+3)$, $\frac{1}{\sin x}$ or $\operatorname{cosec} x$. **4.** $\tan x + c$, $\frac{1}{2}\tan 2x + c$, $\frac{1}{3}\tan(3x+a)+c$. **6.** When x is an odd multiple of $\frac{\pi}{2}$ and dx is finite.

Art. 48. **1.** $-2\csc^2(2x+3)$, $\frac{1}{2}\sec(\frac{1}{2}x+3)\tan(\frac{1}{2}x+3)$, $-3\csc(3x-7)$ $\cot(3x-7)$, $5\sin(5x+2)$, $n\sec^n x\tan x$. **2.** $-6\cot(3t+1)\csc^2(3t+1)$, $\sec^3(\frac{1}{3}t-1)\tan(\frac{1}{3}t-1)$, $-\frac{4}{3}\csc^2\frac{2}{3}(t+5)\cot\frac{2}{3}(t+5)$, $-18t\csc^2 9t^2$, $14(7t-2)\sec(7t-2)^2\tan(7\ \ -2)^2$.

Art. 49. **2.** $\frac{nx^{n-1}}{\sqrt{1-x^{2n}}}$, $\frac{1}{\sqrt{1-2x-x^2}}$, $\frac{2}{1+x^2}$, $\frac{2}{(1-x^2)\sqrt{1-5x^2}}$, $-\frac{1}{\sqrt{1-x^2}}$, $-\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$, $\frac{1}{2}\sqrt{1+\csc x}$. **4.** $\sin^{-1}x + a$, $\sin^{-1}x^2 + a$, $\frac{1}{3}\sin^{-1}x^3 + a$, in which a is an arbitrary constant.

Art. 50. **3.** $\frac{-2nx^{n-1}}{x^{2n}+1}$, $\frac{2}{1+x^2}$, $\frac{a}{\sqrt{2ax-x^2}}$.

Art. 51. **1.** $\frac{2}{1+4x^2}$, $\frac{2}{1+4y^2}\frac{dy}{dx}$, $\frac{2x}{1+x^4}$, $\frac{3y^2}{1+y^6}\frac{dy}{dx}$. **2.** $\frac{4}{1+16t^2}$, $\frac{4t^3}{1+t^8}$, $\frac{6x}{1+9x^4}\frac{dx}{dt}$. **5.** $\frac{2}{1+x^2}$, $\frac{1-x^2}{1+3x^2+x^4}$, $\frac{1}{\sqrt{1-x^2}}$, $\frac{1}{2(1+x^2)}$, $\frac{a}{2(a+2x)\sqrt{x(a+x)}}$, $\frac{3a}{a^2+x^2}$. **7.** $\tan^{-1}x + c$, $\tan^{-1}x^2 + c$, $\frac{1}{4}\tan^{-1}x^4 + c$.

Art. 52. **2.** $\frac{2ax^2}{x^4-a^4}$. **Art. 53.** **2.** $\frac{2}{x\sqrt{x^4-1}}$, $\frac{-2}{\sqrt{1-x^2}}$, $\frac{1}{\sqrt{a^2-x^2}}$, $\frac{-2}{x^2+1}$.

Art. 55. **1.** $\frac{2}{1+x^2}$.

Art. 56. **2.** $(3x^2y^2+3)\,dy + (2xy^3+2)\,dx$, $3(y^2-ax)\,dy + 3(x^2-ay)\,dx$, etc. **3.** $-\sqrt{\frac{y}{x}}$, $-\sqrt[3]{\frac{y}{x}}$, $-\left(\frac{b}{a}\right)^m\left(\frac{x}{y}\right)^{m-1}$, $\frac{y\tan x + \log\sin y}{\log\cos x - x\cot y}$. **4.** $\frac{dx}{2\sqrt{x}}+\frac{dy}{3\sqrt{y}}$, $\frac{2}{3}\left(\frac{dx}{\sqrt[3]{x}}+\frac{dy}{\sqrt[3]{y}}\right)$, $m\left(\frac{x^{m-1}dx}{a^m}+\frac{y^{m-1}dy}{b^m}\right)$, $(y\tan x + \log\sin y)dx - (\log\cos x$ $- x\cot y)\,dy$.

Page 82. **1.** (i) $24x^3 + 15x^2 + 124x + 55$, (ii) $a+b+2x$, (iii) $(a+x)^{m-1}(b+x)^{n-1}[m(b+x)+n(a+x)]$, (iv) $\dfrac{(mx - nx + mb - na)(x+a)^{m-1}}{(x+b)^{n+1}}$, (v) $\dfrac{(m + mx - nx)x^{m-1}}{(1+x)^{n+1}}$, (vi) $\dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$, (vii) $\dfrac{1}{(1+x^2)^{\frac{3}{2}}}$, (viii) $\dfrac{\sqrt{a}(\sqrt{x} - \sqrt{a})}{2\sqrt{x}\sqrt{a+x}(\sqrt{a}+\sqrt{x})^2}$, (ix) $-\dfrac{2}{x^3}\left(1 + \dfrac{1}{\sqrt{1-x^4}}\right)$, (x) $\dfrac{ny}{x\sqrt{1-x^2}}$, (xi) $\dfrac{a^4 + a^2x^2 - 4x^4}{\sqrt{a^2-x^2}}$. **2.** (i) $\dfrac{28x^3 + 6x - 17}{7x^4 + 3x^2 - 17x + 2}$, (ii) $\dfrac{-2a^2x}{a^4 - x^4}$, (iii) $\dfrac{-a}{x\sqrt{a^2-x^2}}$, (iv) $\sec x$, (v) $\dfrac{1}{\sqrt{1+x^2}}$. **3.** (i) $20x^4 \cos 4x^5$, (ii) $-7\sin 14x$, (iii) $6\sec^2 3x \tan 3x$, (iv) $8\sec^2(8x+5)$, (v) $x^{m-1}(1 + m\log x)$, (vi) $pqx^{q-1}\sin^{p-1}x^q \cos x^q$, (vii) $n(\sin x)^{n-1}\sin(n+1)x$, (viii) $\cos(\sin x)\cdot\cos x$, (ix) $\dfrac{\cos(\log nx)}{x}$, (x) $n\cot nx$. **4.** (i) $\dfrac{1}{x^4-1}$, (ii) $\dfrac{2}{\tan^2 x - 1}$, (iii) $\dfrac{x^2}{1-x^4}$. **6.** (i) $\dfrac{1}{e^x + e^{-x}}$, (ii) -1, (iii) $\dfrac{-x}{\sqrt{1-x^2}}$, (iv) $\dfrac{n}{\cos^2 x + n\sin^2 x}$, (v) $\dfrac{-\sqrt{a^2-b^2}}{a + b\cos x}$, (vi) $e^{ax}\sin^{m-1} rx(a\sin rx + mr\cos rx)$, (vii) $-\dfrac{\sec^2 a^{\frac{1}{x}}}{x^2}\log a \cdot a^{\frac{1}{x}}$, (viii) $\dfrac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}$. **7.** (i) $n\left(\dfrac{x}{n}\right)^{nx}\left(1 + \log\dfrac{x}{n}\right)$, (ii) $\dfrac{c}{x}e^{\frac{x}{c}}\left(1 - \dfrac{c}{x}\right)$, (iii) $x^{e^x}e^x\dfrac{1 + x\log x}{x}$, (iv) $e^{x^x}x^x(1 + \log x)$, (v) $x^{(x^x)}\cdot x^x\left\{\dfrac{1}{x} + \log x + (\log x)^2\right\}$, (vi) $x^{x^2+1}(1 + 2\log x)$. **8.** (i) $-\dfrac{ax + hy + g}{hx + by + f}$, (ii) $-\dfrac{x}{y}\cdot\dfrac{2(x^2+y^2) - a^2}{2(x^2+y^2)+a^2}$, (iii) $-\dfrac{2xy^4}{4x^2y^3 + \cos y}$, (iv) $\dfrac{1}{x}\{m\sec(xy) - y\}$, (v) $-\dfrac{\cos x(\cos y + \sin y)}{\sin x(\cos y - \sin y) - 1}$, (vi) $\dfrac{e^x - y}{e^y + x}$, (vii) $\dfrac{y^2 - xy\log y}{x^2 - xy\log x}$, (viii) $\dfrac{my}{x(1+ny)}$. **9.** $\dfrac{\log x}{(1+\log x)^2}$.
10. (i) $2y - \frac{7}{3}$, (ii) $8t - 11$, (iii) $\sec x$, (iv) $-\cot z$, (v) $-\dfrac{\sqrt{1-x^2}}{x}$.
11. (i) $(12x^3 + 18x + 5)(6x^2 + 3)$, (ii) $(e^{\tan t} + 2\tan t)\sec^2 t$, (iii) g, (iv) $\dfrac{e^x x + y}{1 + x^2y^2}$. **12.** (i) 90°, (ii) 73° 41′.2, (iii) 90°, (iv) 2° 21′.7, (v) 70° 31′.7.
14. Speed of Q in inches per second is 116.82, 225, 7, 319.18, 390.9, 436, 451.39, 390.9, 225.7, respectively.

CHAPTER V.

Art. 59. **1.** The lengths of the subnormal, subtangent, tangent, and normal, are respectively: (1) 3, $5\frac{1}{3}$, $6\frac{2}{3}$, 5; (2) 4, 4, 5.66, 5.66; (3) $-\dfrac{b^2x_1}{a^2}$, $-\dfrac{a^2 - x_1^2}{x_1}$, $\dfrac{1}{ax_1}\sqrt{(a^2 - x_1^2)(a^4 - e^2x_1^2)}$, $\dfrac{b\sqrt{a^4 - e^2x_1^2}}{a^2}$, e being the eccentricity; (4) $\sin x_1\cos x_1$, $\tan x_1$, $\tan x_1\sqrt{1+\cos^2 x_1}$, $\sin x_1\sqrt{1+\cos^2 x_1}$; (5) y_1^2, 1,

$\sqrt{1+y_1^2}$, $y_1\sqrt{1+y_1^2}$. **2.** Where x is $7 \pm 2\sqrt{5}$. **3.** Infinitely great. **6.** $xx_1^{-\frac{1}{3}} + yy_1^{-\frac{1}{3}} = a^{\frac{2}{3}}$. **7.** $xx_1^{-\frac{1}{2}} + yy_1^{-\frac{1}{2}} = a^{\frac{1}{2}}$. **8.** $a\sin\theta$, $2a\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}$, $2a\sin\frac{\theta}{2}$, $2a\sin\frac{\theta}{2}\tan\frac{\theta}{2}$. **12.** $90°$, $0°$, $\cot^{-1}4\frac{1}{3}$, *i.e.* $32°\ 12'.5$.

Art. 61. **1.** (1) a, $a\theta^2$, $a\sqrt{1+\theta^2}$, $r\sqrt{1+\theta^2}$; (2) $\frac{a^2}{2r}$, $2r\theta$, $\frac{a}{2}\sqrt{4\theta+\theta^{-1}}$, $a\sqrt{\theta(1+4\theta^2)}$; (3) $-\frac{r^2}{a}$, $-a$, $\frac{r}{a}\sqrt{a^2+r^2}$, $-\sqrt{a^2+r^2}$; (4) $na\theta^{n-1}$, $\frac{a\theta^{n+1}}{n}$, $a\theta^{n-1}\sqrt{n^2+\theta^2}$, $\frac{r}{n}\sqrt{n^2+\theta^2}$. **3.** ar, $\frac{r}{a}$, $r\sqrt{1+a^2}$, $\frac{r}{a}\sqrt{1+a^2}$. **4.** (*a*) $\psi = 84°\ 55'.2$, $\phi = 74°\ 55'.2$; $\psi = 60°\ 41'.9$, $120°\ 41'.9$; (*b*) $\psi = 26°\ 33'.9$, $\phi = 55°\ 12'.8$.

Art. 62. **1.** In feet per second: 0, 4; 2.828, 2.828; 3.57, 1.79; 3.77, 1.33. [Solution for $x = 2$: Where $x = 2$, the tangent to the parabola has a slope 1. Accordingly, the moving point is there going in a direction which is at angle 45° to the x-axis. Hence, the speed of the x-coördinate $\left(i.e.\ \frac{dx}{dt}\right) = \frac{ds}{dt} \times \cos 45° = 4 \times \frac{1}{\sqrt{2}}$; also $\frac{dy}{dt} = 4 \times \frac{1}{\sqrt{2}}$.] **2.** 20 and 22.36 ft. per sec. [Suggestion: Differentiation with respect to the time gives $2y\frac{dy}{dt} = 4\frac{dx}{dt}$.] **3.** .399 and -9.97 ft. per sec.; 9.7 and -2.425 ft. per sec. **4.** 442.82 and 161.6 ft. per sec.; 199.15 and 427.08 ft. per sec. **5.** (1) (2, 8), $(-2, -8)$; (2) $(\frac{1}{6}, \frac{1}{216})$, $(-\frac{1}{6}, -\frac{1}{216})$; (3) 300.

Art. 64. **2.** $\frac{1}{2}$, $\frac{1}{2}$. **3.** The tangent at the middle point of a parabolic arc is parallel to the chord of the arc. **4.** $-3 \pm \sqrt{\frac{37}{3}}$; find the abscissa of the point where the tangent is parallel to the chord joining the points whose abscissas are 3 and 4.

Art. 65. **3.** 25.1 cu. in.; $\frac{1}{10}$. **4.** $4\pi r^2 \cdot \Delta r$; 50.3 sq. in., 502.7 cu. in.; $\frac{1}{100}$, $\frac{2}{100}$, $\frac{3}{100}$. **5.** 1.35 sq. in.; 7 : 5 approximately.

Art. 66. **2.** (1) -1, -1, $\frac{4}{3}$; (2) $-\frac{3}{4}$, $-\frac{3}{4}$, -1; (3) 2, 2, 3, 4; (4) $-\frac{1}{2}$, $-\frac{1}{2}$, 3, $-\frac{5}{2}$; (5) 2, 2, -3, -3, 1. **3.** $n^n r^{n-2} = 4p^n(n-2)^{n-2}$.

Art. 67. **4.** 1.6, .4. **6.** $\frac{1}{2}r^2$, *i.e.* $2\theta^2$; .0048, .035. **7.** .0349, 0, .0025. **9.** $\sqrt{\frac{a+x}{x}}$, $\sqrt{\frac{a+x}{a}}$; $\sqrt[3]{\frac{a}{x}}$. **10.** 2.41, .1. **11.** $a\sqrt{1+\theta^2}$, $\frac{1}{a}\sqrt{a^2+r^2}$. **12.** .078. **14.** πx^4, πx^2. **15.** 5.03, 10.05. **18.** 10.37, 5.06. **19.** $\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}$, $\frac{b}{a}\sqrt{a^2-x^2}$, $\frac{\pi b^2(a^2-x^2)}{a^2}$, $\frac{2\pi b}{a}\sqrt{a^2-e^2x^2}$, e being the eccentricity. **20.** $\frac{a^2}{r}$, a, $r\operatorname{cosec}\alpha$, $\sqrt{2ar}$.

CHAPTER VI.

Art. 68. 1. (i) $\dfrac{2}{(1+x^2)^2}$; (ii) $8+\dfrac{6}{x^3}+\dfrac{1}{4\sqrt{x^3}}$; (iii) $\dfrac{\cos x}{(1-\sin x)^2}$; (iv) $x^x(1+\log x)^2+x^{x-1}$. 2. (i) $\dfrac{4a^3}{(a^2+x^2)^2}$; (ii) $\dfrac{2\cos x}{\sin^3 x}$. 3. (i) $\dfrac{9x+6x^3}{(1-x^2)^{\frac{7}{2}}}$; (ii) $\dfrac{24(1-10x^2+5x^4)}{(1+x^2)^5}$. 4. (i) $-\dfrac{4\,!}{x^2}$; (ii) $-8e^x\sin x$. 6. (i) $-\dfrac{b^4}{a^2y^3}$; (ii) $-\dfrac{24x}{(1+2y)^5}$. 8. (i) -1.4, -2.66; (ii) $\frac{2}{5}$, $-\frac{4}{125}$. 9. $-\dfrac{1}{4a\sin^4\dfrac{\theta}{2}}$; $-\frac{1}{2}$. 12. $24x$. 13. $\dfrac{dy}{dx}=\frac{3}{2}x^2+2x+c_1$, $y=\frac{1}{2}x^3+x^2+c_1x+c_2$, in which c_1 and c_2 are arbitrary constants. 14. $3y=x^3-9x+19$. 15. $y=4x^2+x$. 16. (2) '$-\frac{2}{3}$ ft. per sec.' per sec. 17. In 'in. per sec.' per sec.: (i) $1152\pi^2$, (ii) $768\pi^2$, (iii) $384\pi^2$, (iv) 0. 18. $s=\frac{1}{2}gt^2+c_1t+c_2$. 19. 15.5 sec., 3881.9 ft. 20. $\frac{1}{3000}$ sec.

Art. 69. 2. e^x, $a^x(\log a)^n$, a^ne^{ax}, $b^na^{bx}(\log a)^n$. 4. $\cos\left(x+\dfrac{n\pi}{2}\right)$, $a^n\sin\left(ax+\dfrac{n\pi}{2}\right)$, $a^n\cos\left(ax+\dfrac{n\pi}{2}\right)$. 5. $\dfrac{(-1)^{n-1}(n-1)\,!}{x^n}$, $\dfrac{(-1)^{n-1}2\cdot(n-1)\,!}{(x-2)^n}$.
6. $\dfrac{(-1)^nn\,!}{x^{n+1}}$, $\dfrac{(-1)^nn\,!}{(1+x)^{n+1}}$, $\dfrac{2\cdot n\,!}{(3-x)^{n+1}}$, $\dfrac{(-1)^nac^n(m+n-1)\,!}{(m-1)\,!\,(b+cx)^{m+n}}$.
7. $n\,!\left\{\dfrac{(-1)^n}{(1+x)^{n+1}}+\dfrac{1}{(1-x)^{n+1}}\right\}$, $n\,!\left\{\dfrac{1}{(1-x)^{n+1}}-\dfrac{(-1)^n}{(1+x)^{n+1}}\right\}$.

Art. 71. 2. $\dfrac{a+b\cos\theta}{b\sin\theta}$, $-\dfrac{b+a\cos\theta}{b^2\sin^3\theta}$.

Art. 72. 2. $(x^4-120x^2+120)x\sin x-20(x^2-12)x^2\cos x$. 3. $(x+n)e^x$, $2^{n-1}(n+2x)e^{2x}$.

Art. 73. 3. (1) $y'=xy''$; (2) $x^2y''+2y=2xy'$; (3) $y'+2xy''=0$; (4) $(x^2-2y^2)y'^2-4xyy'-x^2=0$; (5) $yy'=x(yy''+y'^2)$. 4. (1) $y'=0$; (2) $y=xy'$; (3) $y''=0$; (4) $y''=y$; (5) $y''=m^2y$; (6) $y''+m^2y=0$; (7) $y''+m^2y=0$. 5. $y^2(1+y'^2)=r^2$; $x^2(1+y'^2)=r^2y'^2$; $\{1+y'^2\}^{\frac{3}{2}}=ry''$.

CHAPTER VII.

Art. 76. 4. A minimum; neither a maximum nor a minimum. 8. See Ex. 3. 12. See Ex. 3. 13. (1) Min. for $x=\frac{1}{3}$; max. for $x=-2$. (2) Min. at $\dfrac{-1-\sqrt{73}}{6}$; max. at $\dfrac{-1+\sqrt{73}}{6}$. (3) Max. for $x=0$; min. for $x=\dfrac{1-\sqrt{97}}{12}$; min. for $x=\dfrac{1+\sqrt{97}}{12}$; for $x=2$, neither a max. nor a min. (4) Max. for $x=-1$; min. for $x=\frac{1}{5}$; neither a max. nor a min. for $x=2$. (5) Min. for $x=4$. (6) Max. when $x=-4$, and when $x=3$; min. when $x=-3$, and when $x=4$. (7) Min. for $x=16$; max. for $x=4$; neither for $x=10$.

(8) Max. for $x = -10$; min. for $x = -2$; neither for $x = 2$. (9) Min. value is $-\frac{1}{e}$, *i.e.* $-.3678$. (10) Max. when $x = e$. (11) Max. value $= 8$; min. value $= 2$. (12) Max. or min. when $\sin x = \sqrt{\frac{2}{3}}$ according as the angle x is in the first or the second quadrant. (13) Max. when $x = \cot x$. **16.** $(a\sqrt[3]{2},\ a\sqrt[3]{4})$.

Art. 77. **7.** Each factor $= \sqrt{\text{the number}}$. **8.** $\frac{\pi}{2}$. **9.** A square. **10.** (i) $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$; (ii) $a + 2\sqrt{ab} + b$; (iii) $2\,ab$. **11.** Let the perpendiculars drawn from A and B to MN meet MN in R and S respectively; then (1) $RC = CS$; (2) $RC = \frac{AR \cdot RS}{AR + BS}$. **12.** (i) $\frac{4}{3}r$; (ii) $\frac{4}{3}r$. **13.** $19° 28'$. **14.** (i) Vol. $=.5773$ vol. of sphere; (ii) height $= r\sqrt{2}$. **15.** (i) Vol. $= \frac{4}{27}\pi a^2 b$; (ii) height $= \frac{1}{3}b$. **16.** 1. **17.** 2^c, *i.e.* $114° 35' 29''.6$. **18.** $\sqrt{\frac{2}{3}}\,a$. **19.** $1:2$. **22.** $1\frac{1}{2}$ times the rate of the current. **23.** $\frac{\sqrt{3}}{3}d,\ \frac{\sqrt{6}}{3}d$. **24.** $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$. **25.** $\frac{a}{\sqrt{2}}$.

Art. 78. **1.** (1) $(0, 0)$; (2) $(3, -3)$; (3) $(\frac{4}{3}, \frac{128}{27})$; (4) $(2, \frac{5}{3})$; (5) $\left(\pm\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$; (6) where $x = 0$, and where $x = \pm\sqrt{3}$; (7) where $x = 0$, and where $x = \pm 2\sqrt{3}$. **2.** (1) Where $x = \frac{2\,a}{5}$; (2) where $x = \frac{3\,a}{4}$; (3) where $x = \pm\frac{a}{\sqrt{3}}$; (4) (c, b); (5) (c, m); (6) $\left(b, \frac{2\,b^3}{a^2}\right)$.

CHAPTER VIII.

Art. 79. **2.** $3\,x^2 + e^x \sin y$, $4\,y + e^x \cos y - \cos z \sin y$, $6\,z - \sin z \cos y$. **3.** (a) $\frac{-15}{4\sqrt{119}}$ and $\frac{-48}{5\sqrt{119}}$; (b) $\frac{-40}{3\sqrt{89}}$ and $\frac{-24}{5\sqrt{89}}$; (c) $\frac{-20}{3\sqrt{47}}$ and $\frac{-45}{4\sqrt{37}}$, respectively.

Art. 81. **3.** Increasing $\frac{309}{20\sqrt{119}}$ units per second. **4.** Decreasing $\frac{152}{5\sqrt{89}}$ units per second.

Art. 82. **3.** .036; .036011. **4.** (i) $\frac{x\,dy - y\,dx}{x^2 + y^2}$; (ii) $y^x \log y \cdot dx + xy^{x-1}\,dy$; (iii) $yx^{y-1}\,dx + x^y \log x \cdot dy$; (iv) $\frac{y}{x}\,dx + \log x \cdot dy$; (v) $u\left(\frac{\log y}{x}\,dx + \frac{\log x}{y}\,dy\right)$. **5.** .025. **6.** 2.2; 2.37. **7.** .0017. **8.** $x^{yz-1}(yz\,dx + zx \log x \cdot dy + xy \log x \cdot dz)$.

Art. 83. **3.** 4.72 sq. in.

CHAPTER IX.

Art. 90. $\left\{3\left(\frac{d^2x}{dy^2}\right)^2 - \frac{dx}{dy}\frac{d^3x}{dy^3}\right\} \div \left(\frac{dx}{dy}\right)^5$.

Art. 93. **1.** $\{f'(f'\phi''' - \phi' f''') - f''(f'\phi'' - \phi' f'')\} \div f'^5$. **2.** $-4\,a \sin\frac{\theta}{2}$. **3.** $-a$. **4.** $-(a^2 \sin^2\theta + b^2 \cos^2\theta)^{\frac{3}{2}} \div ab$.

Page 147. **1.** (i) $\frac{d^2x}{dy^2} - 2y\frac{dx}{dy} = 0.$ (ii) $\frac{d^3x}{dy^3} + \frac{d^2x}{dy^2} = 0.$ **2.** $\frac{d^2z}{dx^2} - 2\left(\frac{dz}{dx}\right)^2 = \cos^2 z.$ **3.** (i) $\frac{d^2u}{dy^2} + u = 0.$ (ii) $\frac{d^2y}{dt^2} + y = 0.$ (iii) $\frac{d^2y}{dt^2} = 0.$ (iv) $\frac{d^2y}{dz^2} + a^2y = 0.$ (v) $\frac{d^3y}{dz^3} + y = 0.$ (vi) $\frac{d^4y}{dz^4} + 2\frac{d^2y}{dz^2} + y = z.$ **4.** (i) $\tan t$; $\frac{1}{at\cos^3 t}.$ (ii) $-3\sin^4 t\cos t$; $3\sin^5 t(4 - 5\sin^2 t).$

CHAPTER X.

Art. 97. **3.** $y = x^3$; $y = x^3 - 347$; $y = x^3 + 514$; $y - k = x^3 - h^3.$ **4.** $y = 4x + c$; $y = 4x$; $y = 4x - 5$; $y = 4x + 29.$ **5.** $y = 4x^2 + c$; $y = 4x^2$; $y = 4x^2 - 2$; $y = 4x^2 - 13$; $y = 4x - 62.$ **8.** $16t^2$; 64; 256; 400; $16t^2 + 10$, etc.; $16t^2 + 20.$

Art. 98. **3.** $\frac{4}{3}.$ **4.** 2; 0. **5.** 4; 0.

Art. 100. **4.** (*a*) $2y = x^2, 6y = x^3, 24y = x^4$; (*b*) $y = x^2 + 5x, 6y = 2x^3 + 15x^2, 12y = x^4 + 10x^3$; (*c*) $y = 1 - \cos x, y = x - \sin x, 2y = x^2 + 2\cos x - 2$; (*d*) $y = e^x - 1,\ y = e^x - x - 1,\ 2y = 2e^x - x^2 - 2x - 2.$ **5.** $y = 1,\ y = 2,\ y = \cos x,\ y = e^x.$

CHAPTER XI.

Art. 104. **9.** $\frac{1}{3}x^3 + c,\ \frac{6}{73}x^{73} + c,\ \frac{2}{41}x^{41} + c,\ -\frac{2}{9}x^{-18} + c,\ -\frac{5}{13}x^{-13} + c,\ -\frac{1}{2x^2} + c,\ -\frac{3}{x^4} + c,\ \frac{5}{8}x^{\frac{8}{5}} + c,\ \frac{1}{\sqrt{2}+1}x^{\sqrt{2}+1} + c,\ \frac{3}{4}x^{\frac{8}{3}} + c,\ \frac{5}{4}x^{\frac{8}{5}} + c,\ 8\sqrt{x} + c,\ -\frac{10}{\sqrt{x}} + c,\ -\frac{3}{14x^4} + c.$ **10.** $\frac{1}{4}v^4 + c,\ \frac{21}{5}t^{\frac{5}{3}} + c,\ \frac{-1}{2u^4} + c,\ 12s^{\frac{1}{4}} + c.$

11. $\frac{an}{m+n}x^{\frac{m+n}{n}} + c,\ \frac{3c}{m+3}t^{\frac{m+3}{3}} + k,\ \frac{nl}{6+n}v^{\frac{6+n}{n}} + c,\ \frac{rsw^{\frac{t+s}{s}}}{t+s} + c.$ **12.** $\log cv,\ \log c(s+2)^2,\ -\frac{1}{6}\log c(7 - x^6),\ \log c(4t^2 - 3t + 11).$ **13.** $e^t + c,\ \frac{5}{4}e^{4x} + c,\ 2e^{x^2} + c,\ \frac{4^x}{\log 4} + c,\ \frac{10^{2x}}{2\log 10} + c.$ **14.** $-\frac{1}{3}\cos 3x + c,\ \frac{4}{7}\sin 7x + c,\ \frac{9}{5}\tan 5x + c,\ -\cos(x + \alpha) + c,\ \frac{1}{2}\sin(2x + \alpha) + c,\ \frac{5}{3}\tan\left(\frac{3x}{5} + \frac{\pi}{2}\right) + c.$ **15.** $\frac{1}{2}\sec 2x + c,\ \frac{3}{2}\sec\frac{2}{3}x + c,\ \sin^{-1}t + c,\ \frac{1}{2}\sin^{-1}x^2 + c,\ \frac{7}{5}\sin^{-1}5x + c,\ \frac{5}{3}\sin^{-1}x^3 + c,\ \log(v + \sqrt{1 + v^2}) + c,\ \frac{1}{2}\tan^{-1}t^2 + c,\ \tan^{-1}2x + c,\ \sec^{-1}t + c,\ \sec^{-1}3x + c,\ \frac{1}{2}\sec^{-1}x^2 + c,\ \frac{1}{3}\text{vers}^{-1}3x + c$ or $\frac{1}{3}\sin^{-1}(3x - 1) + c,\ \frac{1}{4}\text{vers}^{-1}4x + c$ or $\frac{1}{4}\sin^{-1}(4x - 1) + c.$

16. $\frac{t^5}{5} - \frac{8}{3}t^3 + 16t + c,\quad a^{\frac{9}{4}}x + \frac{12}{7}a^{\frac{3}{2}}x^{\frac{7}{4}} + \frac{6}{5}a^{\frac{3}{4}}x^{\frac{5}{2}} + \frac{4}{13}x^{\frac{13}{4}} + c,\quad \frac{n}{m}e^{\frac{m}{n}x} + c,\ \frac{1}{a}\sin ax - \frac{1}{n}\cos nx + c.$

[In the following integrals the arbitrary constant of integration is omitted.]

Art. 105. **11.** $\frac{1}{6}\sin^6 x,\ \frac{\tan^4 x}{12}(3 + 2\tan^2 x),\ -\frac{1}{7}\tan(4 - 7x),\ -\frac{1}{2}e^{-2x}.$

12. $\log(x+1) + \frac{4x+3}{2(x+1)^2},\ \frac{x^2}{2} + 3x + 3\log x - \frac{1}{x},\ \frac{3}{5}(x+2)^{\frac{2}{3}}(x-8),\ \frac{3}{14}(x-2)^{\frac{4}{3}}$

$(2x+3)$. **13.** $\frac{3}{8}(x+a)^{\frac{8}{3}}$, $\frac{5(m+nx)^{\frac{8}{5}}}{8n}$, $-\frac{2}{7}\sqrt{3-7x}$, $\frac{1}{2}(4+5y)^{\frac{2}{5}}$.
14. $\frac{1}{n}e^{m+nx}$, $-\frac{4^{5-3x}}{3\log 4}$, $\log(\tan^{-1}x)$, $-\cos(\log x)$. **15.** $\frac{2}{15}(t-1)^{\frac{3}{2}}(3t+2)$,
$\frac{3}{5b}(a+by)^{\frac{5}{3}}$, $\frac{4}{7}(m+z)^{\frac{7}{4}}$, $\frac{5}{2}\sin\frac{2}{5}x$. **16.** $\frac{1}{3}\sin x(3-\sin^2 x)$, $\frac{1}{3}\tan x(\tan^2 x+3)$,
$\frac{2}{3}\cos^3 x-\cos x-\frac{1}{5}\cos^5 x$, $n\tan\left(\frac{\theta}{n}\right)$. **17.** $-\frac{1}{7}\log(3+7\cos x)$,
$-\frac{1}{2}\log(9-2\sin x)$, $-\frac{2}{3}\sqrt{4-3\tan x}$, $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{7}}\right)$. **18.** $\sqrt{a^2+x^2}$,
$-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$, $\frac{1}{3}(a^2+x^2)^{\frac{3}{2}}$, $\frac{1}{\sqrt{a^2-x^2}}$.

Art. 106. **7.** $\frac{e^{ax}}{a^2}(ax-1)$. **8.** $-(x+1)e^{-x}$. **9.** $ae^{\frac{x}{a}}(x^2-2ax+2a^2)$.
10. $x\log x-x$. **11.** $\frac{1}{2}x^2(\log x-\frac{1}{2})$. **12.** $\frac{1}{9}x^6(3\log x-1)$. **13.** $x\tan^{-1}x$
$-\log\sqrt{1+x^2}$. **14.** $\frac{1}{2}(1+x^2)\tan^{-1}x-\frac{1}{2}x$. **15.** $2\cos x+2x\sin x-x^2\cos x$.
16. $e^x[x^m-mx^{m-1}+m(m-1)x^{m-2}-\cdots+(-1)^{m-1}m(m-1)\cdots 3\cdot 2\cdot x$
$+(-1)^m\cdot m!]$. **17.** $-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x$. **18.** $-\sqrt{1-x^2}\cdot\sin^{-1}x+x$.

Art. 107. **7.** $\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x+3}{2\sqrt{2}}$; $\sin^{-1}\frac{x-3}{\sqrt{26}}$; $\log(x+3+\sqrt{x^2+6x+10})$.
8. $\frac{1}{8}\log\frac{7+x}{1-x}$; $\sin^{-1}\frac{2x+5}{\sqrt{53}}$; $\log(2x-5+2\sqrt{x^2-5x+7})$. **9.** $\frac{1}{\sqrt{33}}$
$\log\frac{2x+5-\sqrt{33}}{2x+5+\sqrt{33}}$; $\frac{1}{\sqrt{61}}\log\frac{2x+5-\sqrt{61}}{2x+5+\sqrt{61}}$; $\frac{1}{2}\log(8x-3+4\sqrt{4x^2-3x+5})$.
10. $\frac{2}{\sqrt{71}}\tan^{-1}\frac{8x-5}{\sqrt{71}}$; $\frac{1}{2}\sin^{-1}\frac{8x+5}{13}$; $\frac{1}{\sqrt{137}}\log\frac{\sqrt{137}+5+8x}{\sqrt{137}-5-8x}$.
11. $\text{vers}^{-1}\frac{x}{4}$ and $\sin^{-1}\frac{x-4}{4}$; $\frac{1}{2}\text{vers}^{-1}\frac{8x}{9}$ and $\frac{1}{2}\sin^{-1}\frac{8x-9}{9}$; $\frac{1}{25}\sec^{-1}\frac{3x}{5}$.
12. $\frac{1}{2}\sec^{-1}\frac{x-1}{2}$; $\frac{1}{2}\left(x\sqrt{9-x^2}+9\sin^{-1}\frac{x}{3}\right)$; $\frac{25}{4}\pi$. **13.** $x\sqrt{9-x^2}+9\sin^{-1}\frac{x}{3}$;
$\frac{1}{3}\log\tan\left(\frac{3x}{2}+\frac{\pi}{4}\right)$; $\frac{1}{4}\log\tan\frac{4x-\alpha}{2}$. **14.** $\frac{1}{3}\log\sec(3x+\alpha)$; $\frac{1}{8}\log\sin(4x^2+\alpha^2)$;
$\frac{1}{2}\log\tan\left(x+\frac{\pi}{4}\right)$. **16.** $-\frac{(25-x^2)^{\frac{3}{2}}}{75x^3}$; $\frac{x}{4\sqrt{4+x^2}}$; $-\frac{\sqrt{12x-x^2}}{6x}$.

Art. 108. **3.** $\log(x+3)^2+\frac{4}{x-1}$. **5.** $\log(x^2+4)^2(x-1)^7$;
$\log\left(\frac{x^2+4}{x-1}\right)^3-\frac{7}{2}\tan^{-1}\frac{x}{2}$. **6.** $\log\frac{(x-7)^4}{(x+4)^3}$. **7.** $\log(2x+5)(x-7)^3$.
8. $\frac{1}{2}x^2-2x+\log\frac{(x+1)^2}{x-1}$. **9.** $\frac{1}{2}x^2+\log\frac{\sqrt{x^2-1}}{x}$. **10.** $\log\frac{x}{(x-1)^2}+$
$\frac{3}{2}\log(2x+5)$. **11.** $\log\frac{(x-p)(x+q)}{x}$. **12.** $\log(x-3)^2(x+3)^3(x-2)(x+2)^5$.
13. $\log(x-1)-\frac{2}{x-1}$. **14.** $\log\sqrt{4x+5}+\frac{5}{4(4x+5)}$. **15.** $\log x+$

$\frac{3}{2}\log(2x+5)+\frac{2}{x}$. **16.** $\log(x+4)^3\sqrt[3]{3x+2}+\frac{7}{3(3x+2)}$. **17.** $\log(x+1)^2 + \frac{4x+3}{(x+1)^2}$. **18.** $\log x-\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}$. **19.** $\frac{2}{3}\log(3x-2)-\frac{1}{2}\log(x^2+5) - \frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}$. **20.** $\log x+2\tan^{-1}x$. **21.** $x+\frac{1}{6}\log\frac{x^2+3}{x^2}-\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}$.
22. $\log x^2+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}$. **23.** $\log x^3(x^2+3)^2$. **24.** $2\log x-\frac{3}{x}-2\tan^{-1}\frac{x}{2}$.
25. $\log\frac{x-1}{x^2-2x+5}+\frac{1}{2}\tan^{-1}\frac{x-1}{2}$. **26.** $\tan^{-1}x+\log\sqrt{x^2+1}-\frac{3}{x^2+1}$.

Art. 109. **4.** $e^x\cos y$; $x^3+4x^2y+4x-6y$. **5.** $\cos x\tan y-\sin x$; $xe^y-2xy+x^2$; $3x-2x^2-xy-\frac{y^2}{2}$.

Page 190. 1. $\frac{ln^2x^{\sqrt{2}+m+1}}{\sqrt{2}+m+1}+c$, $\frac{1}{2}x^{2(a+b)}+c$, $\frac{r+s}{n+t+3}z^{n+t+3}+c$, $\frac{1}{r^{\frac{1}{3}}s^{\frac{1}{2}}}y^{rs}+c$, $-12\frac{5}{12}+29\log\frac{3}{2}$, $\frac{v^2}{2}+8v-\frac{3}{2}\log(v^2+3)-11\sqrt{3}\tan^{-1}\frac{v}{\sqrt{3}}+c$, $\frac{x^2}{2}-2x+\frac{3}{2}\log(x^2-2)-\frac{5}{2\sqrt{2}}\log\frac{x-\sqrt{2}}{x+\sqrt{2}}+c$, $\frac{7}{6\sqrt{5}}\tan^{-1}\frac{3t}{2\sqrt{5}}+c$, $\frac{1}{4\sqrt{3}}\log\frac{z-2\sqrt{3}}{z+2\sqrt{3}}+c$, $7a^{\frac{4}{3}}+\frac{186}{5}a^{\frac{2}{3}}+\frac{381}{7}$, $\frac{1}{3}\sin^{-1}\frac{x^3}{3}+c$, $\frac{1}{3}\log(x^3+\sqrt{x^6-9})+c$, $\frac{1}{4}z+\frac{3}{8(2z-1)}+\frac{1}{4}\log(2z-1)+c$. **2.** $\frac{1}{m}\log\sec(mx+n)+c$, $\frac{1}{3}\tan 3x+\frac{4}{3}\log(\sec 3x+\tan 3x)+4x+c$, ∞, 2.4288.
3. $x\cos^{-1}x-\sqrt{1-x^2}+c$, $x\sec^{-1}x-\log(x+\sqrt{x^2-1})+c$, $x\cot^{-1}x+\frac{1}{2}\log(1+x^2)+c$, $x\{(\log x)^2-2\log x+2\}+c$, $-ae^{-\frac{x}{a}}(x^2+2ax+2a^2)+c$, $-(x^3+3x^2+6x+6)e^{-x}+c$, $\cos x(1-\log\cos x)+c$, $\frac{x^{m+1}}{m+1}\left(\log x-\frac{1}{m+1}\right)+c$. **4.** $\frac{6}{5}x^{\frac{5}{6}}-\frac{4}{3}\sqrt{x}+c$, $18(\frac{1}{7}x^{\frac{7}{6}}+\frac{1}{5}x^{\frac{5}{6}}+\frac{1}{3}x^{\frac{1}{2}}+x^{\frac{1}{6}})+9\log\frac{x^{\frac{1}{6}}-1}{x^{\frac{1}{6}}+1}+c$, $4(3^{\frac{1}{4}}-2^{\frac{1}{4}})+4\log\frac{3^{\frac{1}{4}}-1}{2^{\frac{1}{4}}-1}$, $\sqrt{x^2-1}+\log(x+\sqrt{x^2-1})+c$. **5.** .206 (the base being 10), $\frac{1}{3}\left(1-\frac{1}{e^3}\right)$, $\frac{1}{3}(e^3-1)$, $-\frac{19}{648}\pi^3$. **6.** $-\frac{a}{n}\log(m+n\cos\theta)+c$, $\log\left(\sin\theta\tan\frac{\theta}{2}\right)+c$, $\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$, $-\frac{\sec x}{8(\sec^2x-4)}-\frac{1}{32}\log\frac{\sec x-2}{\sec x+2}+c$ (see result in Ex. 3, Art. 118), $\sin^{-1}\left(\frac{\tan\theta}{2}\right)+c$, $\frac{1}{3m}\log^3(mz+n)+c$, $\frac{1}{m\log a}\sec^{-1}\frac{a^x}{m}+c$, $\tan^{-1}e^x+c$, $\frac{1}{4}\log\frac{e^x-e^{-x}}{e^x+e^{-x}}+c$, $\frac{1}{4}\log\frac{1+\tan 2\theta}{1-\tan 2\theta}+c$, $4\sqrt{2}\sin^{-1}\left(\sqrt{2}\sin\frac{x}{4}\right)+c$, $\cos x\cos y-y^2+x+c$, $\cos x\sin y+x-y+c$.

CHAPTER XII.

Art. 111. **5.** (*b*) 76. **6.** 18. **8.** 5. **11.** $\frac{20}{3}\sqrt{5}$. **13.** (*a*) 2; (*d*) 4. **16.** .862025; 6.644025; .862; .401. **17.** (1) $\frac{1}{32}$; (2) $10\frac{2}{3}$; (3) 3.2; (4) $68\frac{4}{15}$; (5) $\frac{1}{6}a^2$; (6) $12\sqrt{2}$; (7) No area is bounded; (8) (*a*) log 7, *i.e.* 1.946; log 15, *i.e.* 2.708; $\log n$; $k^2\log\frac{b}{a}$. **18.** $\frac{144}{35}\sqrt{\frac{3}{2}}$.

Art. 112. **9.** $\frac{512}{15}\pi$. **10.** $\frac{1024}{5}\pi$. **11.** $\frac{768}{5}\pi$. **12.** (*a*) $\frac{6}{5}(2\sqrt[3]{4}-1)\pi$; (*b*) $\frac{4}{7}(4\sqrt[3]{2}-1)\pi$. **13.** $\frac{243}{80}\pi$. **18.** $405\left(\frac{5}{3}-\frac{\pi}{2}\right)\pi$, $225\left(\frac{5}{3}-\frac{\pi}{2}\right)\pi$.

Art. 113. **2.** $y^2=48x-80$; 24. **3.** $x-4=2\log y$. **4.** $x-4=4\log y$; 4. **5.** $3y^2=16x$. **6.** $5y^2=48x^2-112$; the conics $y^2=kx^2+c$, k and c denoting arbitrary constants. **7.** $3y=x^2+6$; the parabolas $y=kx^2+c$, k and c being arbitrary constants. **8.** $y^2=7x+4$; the parabolas $y^2=kx+c$, k and c being any constants. **9.** The circles $r=c\sin\theta$; $r=4\sin\theta$. **10.** $r^2=ce^\theta$; $r^2=4e^\theta$. **11.** $r=a(1-\cos\theta)$, in which a is an arbitrary constant.

CHAPTER XIII.

Art. 116. **1.** $\frac{4}{3}\sqrt[4]{x}(\sqrt{x}-3)+4\tan^{-1}\sqrt[4]{x}+c$. **2.** $2(\sqrt{x}-\tan^{-1}\sqrt{x})+c$.
3. $\frac{1}{6}(3x-2)^{\frac{2}{3}}-\dfrac{2}{3\sqrt[3]{3x-2}}+c$. **4.** $\frac{2}{35}(2+x)^{\frac{5}{2}}(5x+17)+c$.
5. $-\frac{2}{5}\log(7+5\sqrt{2-x})+c$. **6.** $x+1+4\sqrt{x+1}+4\log(\sqrt{x+1}-1)+c$.

Art. 117. **5.** $\frac{1}{2}\sqrt{4x^2+6x+11}+\frac{7}{4}\log(2x+3+\sqrt{4x^2+6x+11})+c$.
6. $-3\sqrt{12-4x-x^2}-10\sin^{-1}\dfrac{x+2}{4}+c$. **7.** $\dfrac{1}{2\sqrt{3}}\log\dfrac{\sqrt{6-3x}-\sqrt{6+x}}{\sqrt{6-3x}+\sqrt{6+x}}+c$.
8. $3\sin^{-1}\dfrac{x+2}{4}-\dfrac{2}{\sqrt{3}}\log\dfrac{\sqrt{6-3x}-\sqrt{6+x}}{\sqrt{6-3x}+\sqrt{6+x}}+c$. **9.** $\log\dfrac{x-1+\sqrt{x^2+x+1}}{x+1+\sqrt{x^2+x+1}}+c$.
10. $\sqrt{x^2+x+1}+\frac{3}{2}\log(x+\frac{1}{2}+\sqrt{x^2+x+1})-3\log\dfrac{x-1+\sqrt{x^2+x+1}}{x+1+\sqrt{x^2+x+1}}+c$.
11. $\frac{1}{4}\sec^{-1}\dfrac{x+2}{4}+c$.

Art. 120. **2.** $\frac{1}{3}\cos^3x-\cos x+c$; $\sin x-\frac{1}{3}\sin^3x+c$; $\frac{2}{3}\cos^3x-\frac{1}{5}\cos^5x-\cos x+c$. **4.** (1) $\frac{3}{8}\cos^{\frac{2}{3}}x(\cos^2x-4)+c$; (2) $5\sin^{\frac{8}{5}}x(\frac{1}{8}-\frac{1}{9}\sin^2x+\frac{1}{28}\sin^4x)+c$; (3) $2\sqrt{\sin x}(1-\frac{2}{5}\sin^2x+\frac{1}{9}\sin^4x)+c$; (4) $3\cos^{\frac{5}{3}}x(\frac{1}{11}\cos^2x-\frac{1}{5})+c$.
7. (1) $\frac{1}{3}\tan^3x+\tan x+c$; (2) $-\frac{1}{3}\cot^3x-\cot x+c$; (3) $\frac{1}{5}\tan^5x+\frac{2}{3}\tan^3x+\tan x+c$.
9. (1) $\frac{1}{15}\tan^3x(3\tan^2x+5)+c$; (2) $2\tan^{\frac{3}{2}}x(\frac{1}{3}+\frac{2}{7}\tan^2x+\frac{1}{11}\tan^4x)+c$; (3) $\frac{3}{2}\tan^{\frac{8}{3}}x(\frac{1}{4}+\frac{1}{7}\tan^2x)+c$; (4) $\sec^3x(\frac{1}{7}\sec^4x-\frac{2}{5}\sec^2x+\frac{1}{3})+c$; (5) $\frac{2}{5}\sqrt{\csc x}(5-\csc^2x)+c$; (6) $-\csc^3x(\frac{1}{7}\csc^4x-\frac{2}{5}\csc^2x+\frac{1}{3})+c$.

Art. 121. **3.** (1) $\frac{1}{4}\left(\frac{3x}{2}-\sin 2x+\frac{\sin 4x}{8}\right)+c$; (2) $\frac{1}{16}(5x+4\sin 2x - \frac{4}{3}\sin^3 2x+\frac{3}{4}\sin 4x)+c$; (3) $\frac{x}{16}-\frac{\sin 4x}{64}-\frac{\sin^3 2x}{48}+c$; (4) $\frac{1}{48}\cos 2x(\cos^2 2x-3)+c$; (5) $\frac{1}{128}\left(3x-\sin 4x+\frac{\sin 8x}{8}\right)+c$.

Art. 122. **1.** (1) $-\frac{\sin x\cos x}{2}+\frac{x}{2}+c$; (2) $-\frac{1}{3}\sin^2 x\cos x-\frac{2}{3}\cos x+c$; (3) $-\frac{\cos x\sin x}{4}(\sin^2 x+\frac{3}{2})+\frac{3}{8}x+c$; (4) $-\frac{1}{5}\sin^4 x\cos x-\frac{4\cos x}{15}(\sin^2 x+2)+c$.
2. (1) $-\cot x+c$; (2) $\log\tan\frac{x}{2}-\frac{1}{2}\cot x\csc x+c$; (3) $-\frac{1}{3}\frac{\cos x}{\sin^3 x}-\frac{2}{3}\cot x+c$.
5. (1) $\frac{1}{8}\sin x\cos x(2\cos^2 x+3)+\frac{3}{8}x+c$; (2) $\frac{1}{5}\sin x(\cos^4 x+\frac{4}{3}\cos^2 x+\frac{8}{3})+c$; (3) $\frac{1}{3}\frac{\sin x}{\cos^3 x}+\frac{2}{3}\tan x+c$; (4) $\frac{1}{4}\tan x\sec^3 x+\frac{3}{8}\sec x\tan x+\frac{3}{8}\log(\sec x+\tan x)+c$.
6. (1) $\frac{1}{2}\tan x\sec x+\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$; (2) $\frac{1}{3}\tan x(\sec^2 x+2)+c$; (3) $\frac{1}{4}\tan x\sec^3 x+\frac{3}{8}\left\{\tan x\sec x+\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}+c$. **7.** (1) $\frac{1}{2}\log\tan\frac{x}{2} - \frac{1}{2}\cot x\,\text{cosec}\, x+c$; (2) $-\frac{1}{3}\cot x(\text{cosec}^2 x+2)+c$; (3) $-\frac{1}{4}\cot x\,\text{cosec}^3 x - \frac{3}{8}\left(\cot x\,\text{cosec}\, x-\log\tan\frac{x}{2}\right)+c$. **11.** (1) $\frac{1}{2}\tan^2 x-\log\sec x+c$; (2) $-\frac{1}{3}\cot^3 x+\cot x+x+c$; (3) $\frac{1}{3}\tan^3 x-\tan x+x+c$; (4) $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x +\log\sec x+c$. **14.** (1) $\frac{1}{8}(\sin x\cos x+x)-\frac{1}{4}\sin x\cos^3 x+c$; (2) $-\frac{1}{6}\sin x\cos^5 x +\frac{1}{24}\sin x\cos^3 x+\frac{1}{16}\sin x\cos x+\frac{1}{16}x+c$; (3) $-\frac{1}{2}\frac{\cos x}{\sin x}(3-\cos^2 x)-\frac{3x}{2}+c$.
17. (1) $-\frac{1}{7}\cot^7 x-\frac{1}{5}\cot^5 x+c$; (2) $\frac{1}{4}\tan^4 x+c$; (3) $-\frac{1}{15}\cot^3 x(3\cot^2 x+5)+c$.

Page 222. **3.** (1) $3t^{\frac{1}{3}}+\frac{1}{2}\log\frac{(t^{\frac{1}{3}}-1)^3}{t-1}-\sqrt{3}\tan^{-1}\left(\frac{2t^{\frac{1}{3}}+1}{\sqrt{3}}\right)+c$; (2) $\frac{3(2v+3)}{8\sqrt[3]{2v+1}}+c$; (3) $\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}x}{\sqrt{1-4x^2}}\right)+c$; (4) $\frac{1}{2\sqrt{5}}\log\frac{\sqrt{1-4x^2}-\sqrt{5}}{\sqrt{1-4x^2}+\sqrt{5}}+c$; (5) $-\frac{2\sqrt{4x-x^2}}{x}-\text{vers}^{-1}\frac{x}{2}+c$; (6) $2\sqrt{x^2+3x+5}-2\log(x+\frac{3}{2}+\sqrt{x^2+3x+5})+c$; (7) $2\log(x+\frac{3}{2}+\sqrt{x^2+3x+5})+\frac{1}{\sqrt{5}}\log\frac{10+3x-2\sqrt{5(x^2+3x+5)}}{x}+c$; (8) $-\frac{1}{\sqrt{2}}\log\frac{1-v+\sqrt{2(v^2+1)}}{v+1}+c$; (9) $-\frac{1}{128}\left\{\frac{x^2}{(x^2-16)^2}-\frac{3x^2}{32(x^2-16)} + \frac{3}{256}\log\frac{x^2-4}{x^2+4}\right\}+c$; (10) $\frac{x(3x^2+20)}{128(x^2+4)^2}+\frac{3}{256}\tan^{-1}\frac{x}{2}+c$; (11) $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)+c$; (12) $\cos^{-1}\left(\frac{x-3}{3}\right)-2\sqrt{\frac{6-x}{x}}+c$.

CHAPTER XIV.

Art. 125. **2.** .2525. **3.** 3690; 3660; (true value = 3660). **6.** 333 in 20,000. **7.** .05075; 1509.

CHAPTER XV.

Art. 129. **4.** The parabolas $y = 3x^2 + c_1x + c_2$, whose axes are parallel to the y-axis; $2y = 6x^2 + 11x - 13$; $y = 3x^2 + 15x + 22$. **5.** The cubical parabolas $y = x^3 + c_1x + c_2$; $y = x^3 + x$; $y = x^3 - x + 4$. **6.** The cubical parabolas $y = cx^3 + c_1x + c_2$, in which c, c_1, c_2 are arbitrary constants; $6y = x^3 + 11x$; $5y + x^3 + 16 = 22x$. **7.** The cubical parabolas $x = c_1y^3 + c_2y + c_3$; $120x = 11y^3 - 251y + 240$; $7x + 4y^3 = 62y - 85$. **8.** 15,528 ft.; 62.1 sec. **10.** Half a mile.

Art. 130. **4.** (1) 37; (2) $38\frac{1}{15}a^3$; (3) $6a^3$; (4) $-\frac{8}{3}a^3\pi$; (5) $\frac{1}{6}\pi abc$; (6) $\frac{4}{3}\pi a^3$; (7) $\frac{\pi a^2}{2}$; (8) $\frac{\pi a^2}{2}$; (9) $\frac{1}{6}\pi a^3 - \frac{2}{9}a^3$.

Art. 131. **3.** 5.

Art. 132. **5.** 1154.7 cu. in. **6.** $\frac{2}{3}a^3\tan\alpha$. **7.** $\frac{4}{3}(\pi - \frac{4}{3})a^3$. **8.** 2720.3 cu. in.; $\frac{\pi a^3}{2}\tan\alpha$.

Art. 133. **4.** $\frac{4}{3}\pi(a^2 - b^2)^{\frac{3}{2}}$.

CHAPTER XVI.

Art. 135. **4.** 301.6; $\frac{1}{2}\pi abh$. **5.** $55\frac{5}{9}$ cu. ft. **6.** $\frac{4}{3}ab^2\cot\alpha$. **7.** $\frac{2}{3}(3\pi + 8)a^3$. **8.** $\frac{4}{3}a^2h$.

Art. 136. **2.** $\frac{\pi a^2}{12}$. **3.** $\frac{a^2}{2}$; $\frac{a^2}{n}$. **5.** $\frac{3}{8}\pi a^2$. **6.** 11π. **7.** $\frac{3}{2}a^2$.

Art. 137. **2.** (1) $2\pi a$; (2) (b) $\{\sqrt{2} + \log(\sqrt{2} + 1)\}a$; (3) $4a\left(\cos\frac{\theta_0}{2} - \cos\frac{\theta_1}{2}\right), 8a$; (4) $\frac{a}{2}(e^{\frac{x_1}{a}} - e^{-\frac{x_1}{a}}), \frac{a}{2}\left(e - \frac{1}{e}\right)$. **3.** $\frac{4(a^2+ab+b^2)}{a+b}$.

Art. 138. **2.** (3) $\frac{3\pi a}{2}$; (4) (a) $l\sec\alpha$, in which l is the difference in length of the radii vectores to the extremities of the arc; (4) (b) like (4) (a); (5) $\frac{a}{2}\left[\theta_2\sqrt{1+\theta_2^2} - \theta_1\sqrt{1+\theta_1^2} + \log\frac{\theta_2 + \sqrt{1+\theta_2}}{\theta_1 + \sqrt{1+\theta_1}}\right]$; (6) $a\tan\frac{\theta_1}{2}\sec\frac{\theta_1}{2} + a\log\tan\left(\frac{\theta_1}{4} + \frac{\pi}{4}\right)$; $2a\left(\sec\frac{\pi}{4} + \log\tan\frac{3}{8}\pi\right)$.

Art. 139. **5.** $4\pi a^2$. **6.** $\pi(\pi - 2)a^2$. **7.** $2\pi b^2 + 2\pi ab\frac{\sin^{-1}e}{e}$. **8.** (1) $3\pi a^2$; (2) $5\pi^2a^3$, $\frac{64}{3}\pi a^2$; (3) π^2a^3, $\frac{32}{3}\pi a^2$. **9.** $2\pi^2a^2b$, $4\pi^2ab$. **10.** $2\pi a^2\left(1 - \frac{1}{e}\right)$. **12.** $\frac{1}{24}\pi a^3(22 + 3\pi)$; $\frac{1}{2\sqrt{2}}\pi a^2(\pi + 4)$.

Art. 140. **2.** $4a^2$. **3.** $4\pi a^2$. **4.** Surface $= 8a\left(2b\sin^{-1}\frac{b}{\sqrt{a^2 - b^2}} - a\sin^{-1}\frac{b^2}{a^2 - b^2}\right)$.

Art. 141. **3.** $134\frac{1}{3}$; $9\frac{1}{4}$. **4.** 4.64. **5.** (1) $2\frac{2}{3}$, $5\frac{1}{3}$; (2) $\frac{3}{4}$, 1.14, .94; (3) $5\frac{1}{2}$, $9\frac{1}{2}$. **6.** (1) 9.425; (2) 15.71; (3) $1.571\,b$, $1.571\,a$. **7.** $\frac{2b}{\pi}$, $\frac{2a}{\pi}$. **9.** $\frac{1}{6}n^2$. **10.** $1.273\,a$. **12.** $1.132\,a$, $1.5\,a^2$. **13.** $\frac{2}{3}a$, $\frac{1}{2}a^2$. **14.** 32.704°. **15.** $\frac{1}{2}a$, $\frac{2}{3}a$. **16.** $\frac{6}{5}a$, $\frac{8}{5}a^2$. **17.** $\frac{3}{4}a$, $\frac{3}{5}a^2$. **18.** $1.273\,a$, $2\,a^2$. **19.** $.6366\,a$, $\frac{1}{2}a^2$.

CHAPTER XVII.

Art. 143. **1.** (1) First order at (1, 1); (2) second order at (2, 8). **2.** $y = 5x^2 - 6x + 3$. **3.** -1. **4.** $y = 3x^2 - 3x + 1$. **5.** $y = x^2 - 3x + 3$.

Art. 144. **1.** 5.27 and $(-4, \frac{4}{3})$; 2.635 and $(-\frac{3}{2}, \frac{11}{6})$. **2.** $R = 145.5$; $(-143, 20\frac{1}{12})$.

Art. 148. **1.** The curvature of $y=x^3$ is one-half that of $y = 6x^2 - 9x + 4$. **2.** $\frac{\sqrt{2}}{125}$; $R = -88.4$; $(-87.5, -12.5)$.

Art. 149. **3.** $\frac{2(p+x)^{\frac{3}{2}}}{p^{\frac{1}{2}}}$; $\left(2p + 3x, -\frac{y^3}{4p^2}\right)$; $2p$ and $(2p, 0)$.
4. $R = -\frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{a^4b^4} = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}$; Centre at $\left(\frac{a^2 - b^2}{a^4}x^3, -\frac{a^2 - b^2}{b^4}y^3\right)$.
5. (1) $R = \frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{a^4b^4} = \frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}$; $\left(\frac{a^2 + b^2}{a^4}x^3, -\frac{a^2 + b^2}{b^4}y^3\right)$.
(2) $\frac{(x^2 + y^2)^{\frac{3}{2}}}{2a^2}$; $\left(\frac{3}{2}x + \frac{y^3}{2a^2}, \frac{3}{2}y + \frac{x^3}{2a^2}\right)$. (3) $\frac{y^2}{a}$; $\left(x - \frac{y\sqrt{y^2 - a^2}}{a}, 2y\right)$.
(4) $-3(axy)^{\frac{1}{3}}$; $(x + 3\sqrt[3]{xy^2}, y + 3\sqrt[3]{x^2y})$. (5) $3a\sin\theta\cos\theta$; $(a\cos^3 t + 3a\cos t\sin^2 t, a\sin^3 t + 3a\sin t\cos^2 t)$. (6) $\frac{(4a + 9x)^{\frac{3}{2}}x^{\frac{1}{2}}}{6a}$; $\left(-x - \frac{9}{2}\frac{x^2}{a}, 4y + \frac{4}{3}\frac{ay}{x}\right)$. (7) $-2a$; $(a, -\frac{3}{2}a)$. (8) $\pm 4a\sin\frac{\theta}{2}$; $(a \cdot \overline{\theta + \sin\theta}, -y)$.
6. (1) $\frac{(x + y)^{\frac{3}{2}}}{2\sqrt{a}}$. (2) $\frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^4x}$. (3) $c\sec\frac{x}{c}$. (4) $\frac{1}{2}a$. (5) $2a\,\mathrm{cosec}^3\psi$.
(6) $\frac{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}}{ab}$. (7) $-\frac{(a^2\tan^2\phi + b^2\sec^2\phi)^{\frac{3}{2}}}{ab}$. (8) $a\sec^2\theta$, *i.e.* $\frac{y^2}{a}$.

Art. 150. **1.** (1) a; b. (2) $\frac{2r\sqrt{r}}{\sqrt{a}}$. (3) $\frac{2}{3}\sqrt{2ar}$. (4) $-\frac{r^3}{a^2}$. (5) $\pm\frac{a^2}{3r}$. (6) $r\sqrt{1 + a^2}$. (7) $\pm\frac{a(1 + \phi^2)^{\frac{3}{2}}}{2 + \phi^2}$. (8) $\pm\frac{a\phi^{n-1}(n^2 + \phi^2)^{\frac{3}{2}}}{n(n + 1) + \phi^2}$.

Art. 151. **3.** (1) $(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$. (2) $(x + y)^{\frac{2}{3}} - (x - y)^{\frac{2}{3}} = (4a)^{\frac{2}{3}}$. [Suggestion: Show that $\alpha + \beta = \frac{a}{2}\left(\frac{a}{x} + \frac{x}{a}\right)^3$, $\alpha - \beta = \frac{a}{2}\left(\frac{a}{x} - \frac{x}{a}\right)^3$, and deduce therefrom.] (3) $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

CHAPTER XVIII.

Art. 157. **5.** (1) $x^2 + y^2 = a^2$. (2) $b^2x^2 + a^2y^2 = a^2b^2$. (3) $4ay^2 + bxy + cx^2 = 4ac - b^2$. (4) $4xy + a^2 = 0$. (5) $4y^3 = 27a^2x$. (6) $(x-a)^2 + (y-b)^2 = r^2$. **6.** (1) $x^2 + y^2 = a^2$. (2) $x^2 - y^2 = a^2$. (3) $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$. **7.** (1) The lines $x \pm y = 0$; (2) $27cy^2 = 4x^3$. **10.** A parabola; $y^2 = 4ax$ if the fixed point be $(a, 0)$ and the fixed line be the y-axis.

Art. 158. **3.** $4xy = a^2$. **4.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. **5.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Art. 160. **4.** (1) $x = a, y = b$. (2) $x = 2$. (3) $y + 3 = 0, 2x + 3 = 0$. (4) $y + 1 = 0$. **5.** (2) $(2, \frac{9}{4})$. (3) $(-\frac{7}{4}, -3), (-\frac{3}{2}, -\frac{11}{5})$. (4) $(-\frac{1}{3}, -1)$. **8.** (1) $x=0, y=0$. (2) $x=2a$. (3) $y=0$. (4) $x=\pm a, y=\pm b$. (5) $y=0, x=a$. (6) $x=0$. (7) $y=0$. (8) $y=0$. (9) $x=(\pm 2n+1)\frac{\pi}{2}$, in which n is any integer.

Art. 161. **2.** $bx \pm ay = 0$. **5.** (1) $y = x$. (2) $x + y = 1, x - y = 1$. (3) $x=2, y+3=0, 2(y-x)=5$. (4) $x=y\pm 1, x+y=\pm 1$. (5) $6y=3x+2$.

Art. 162. **2.** (1) Lines parallel to the initial line and at a distance $\pm na\pi$ from it, n being any integer. (2) The line perpendicular to the initial line, at a distance a to the left of the pole. (3) The two lines which are parallel to the initial line and are at a distance $2a$ from it. **4.** $r\sin(\theta - 1) = 1$; $r = 1$.

Art. 165. **3.** (1) Node at origin; slopes there are ± 1. (2) Cusp at $(-3, 1)$; slope there is 0. (3) Cusp at $(2, 1)$; tangent there is parallel to the y-axis. (4) Double point at $(0, 0)$; slopes of tangents there are $1, -\frac{3}{2}$. (5) Cusp at $(1, 2)$; slope of tangent there is 1. (6) A conjugate point at $(3, 0)$.

CHAPTER XIX.

Art. 171. **3.** (1) Convergent. (2) Convergent. (3) Divergent. (4) Divergent except when $p > 2$. (5) Convergent if $p > 2$. **4.** (1) $x < 1$, convergent; $x > 1$, or $x = 1$, divergent. (2) Absolutely convergent if $x^2 < 1$, divergent if $x^2 = 1$, divergent if $x^2 > 1$. (3) Absolutely convergent for all values of x. (4) $x < 1$, or $x = 1$, convergent; $x > 1$, divergent. (5) Same as in Ex. 4.

CHAPTER XX.

Art. 176. **5.** (a) $\cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x + \cdots$; (b) $\cos x - x\sin h - \frac{x^2}{2!}\cos h + \frac{x^3}{3!}\sin h + \cdots$.

Art. 177. **4.** $e + e(x-1) + \frac{e}{2!}(x-1)^2 + \cdots$.

Art. 178. **10.** (1) $1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+\cdots$; (2) $\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots$. **12.** (1) $c+x+\frac{x^2}{2}-\frac{2x^4}{4!}-\frac{2^2\cdot x^5}{5!}-\frac{2^2\cdot x^6}{6!}+\frac{2^3\cdot x^8}{8!}+\cdots$; (2) $\log\frac{b}{a}+(b-a)+\frac{b^2-a^2}{2\cdot 2!}+\frac{b^3-a^3}{3\cdot 3!}+\cdots$; (3) $x-\frac{x^3}{1\cdot 3}+\frac{x^5}{1\cdot 2\cdot 5}-\frac{x^7}{1\cdot 2\cdot 3\cdot 7}+\cdots$.

CHAPTER XXI.

Art. 186. **2.** $y\sqrt{1-x^2}+x\sqrt{1-y^2}=c$. **3.** $(y+b)^n(x+a)^m=c$.

Art. 187. **1.** $x^2+y^2=cy$. **2.** $x^2(x^2+2y^2)=c^4$. **3.** $xy^2=c^2(x+2y)$. **4.** $xy(x-y)=c$.

Art. 188. **1.** $xy=c$. **2.** $x^2y+3x+2y^2=c$. **3.** $e^x\sin y+x^2=c$. **4.** $3axy-y^3=x^3+c$. **7.** $a\log(x^2y)-y=c$. **8.** $\log\frac{cx}{y}=\frac{1}{xy}$.

Art. 189. **3.** $\sqrt{1-x^2}\cdot y=\sin^{-1}x+c$. **4.** $y=\tan x-1+ce^{-\tan x}$. **5.** $y=x^2(1+ce^{\frac{1}{x}})$. **7.** $3y^{\frac{1}{3}}=c(1-x^2)^{\frac{1}{3}}-1+x^2$. **8.** $y^2(x^2+1+ce^{x^2})=1$.

Art. 190. **2.** $y^2=2cx+c^2$. **3.** $y=c-[p^2+2p+2\log(p-1)]$, $x=c-[2p+2\log(p-1)]$. **4.** $\log(p-x)=\frac{x}{p-x}+c$, with the given relation. **5.** $(x^2+y)^2(x^2-2y)+2x(x^2-3y)c=c^2$. **6.** $y=cx+\frac{a}{c}$. **7.** $y=cx+a\sqrt{1+c^2}$. **8.** $y^2=cx^2+c^2$.

Art. 191. **2.** $x^2+y^2=a^2$; $x^2(x^4-4y^2)=0$. **3.** (1) $y=cx+c^2$, $x^2+4y=0$. (2) $(y+x-c)^2=4xy$, $xy=0$. (3) $(x-y+c)^3=a(x+y)^2$, $x+y=0$.

Art. 192. **3.** The concentric circles $x^2+y^2=a^2$. **4.** The lines $y=mx$. **8.** (1) The ellipses $y^2+2x^2=c^2$; (2) the hyperbolas $x^2-y^2=c^2$; (3) the conics $x^2+ny^2=c$; (4) the curves $y^{\frac{4}{3}}-x^{\frac{4}{3}}=c^{\frac{4}{3}}$; (5) the ellipses $x^2+2y^2=c^2$; (6) the cardioids $r=c(1+\cos\theta)$; (7) the curves $r^n\cos n\theta=c^n$; (8) the curves $r^n=c^n\sin n\theta$; (9) the lemniscates $r^2=c^2\sin 2\theta$, whose axes are inclined at an angle 45° to the axes of the given system; (10) the confocal and coaxial parabolas $r(1-\cos\theta)=2c$; (11) the circles $x^2+y^2-2lx+a^2=0$, in which l is the parameter. **10.** The conics that have the fixed points for foci. **11.** The conics that have the fixed points for foci. **12.** The conics $b^2x^2\pm a^2y^2=a^2b^2$. **13.** The hyperbola $4xy=a^2$. **14.** The parabola $(x-y)^2-2a(x+y)+a^2=0$.

Art. 193. **3.** (1) $y=e^{2x}(a\cos 3x+b\sin 3x)$. (2) $y=c_1e^{2x}+c_2e^x+c_3e^{3x}$. (3) $y=c_1e^{4x}+e^{-2x}(c_2+c_3x)$. (4) $y=e^{2x}(c_1+c_2x)+e^{3x}(c_3\cos 5x+c_4\sin 5x)$. **7.** (1) $y=x(a\cos\log x+b\sin\log x)$. (2) $y=x(c_1+c_2\log x)$. (3) $y=x^2(c_1+c_2\log x)$. (4) $y=c_1x^{-1}+x(c_2\cos\log x+c_3\sin\log x)$. **9.** $y=(5+2x)^2\{c_1(5+2x)^{\sqrt{2}}+c_2(5+2x)^{-\sqrt{2}}\}$.

Art. 194. **4.** (1) $y = c_1 e^{ax} + c_2 e^{-ax}$. (2) $e^{2cx} + 2cc_1 e^{cx-y} = c_1^2$. (3) $t = \sqrt{\dfrac{a}{2k}} \left\{ \dfrac{a}{2} \left(\text{vers}^{-1} \dfrac{2x}{a} - \pi \right) - \sqrt{ax - x^2} \right\}$. **5.** The circle of radius a. **6.** (1) $y = c_1 x + (c_1^2 + 1) \log(x - c_1) + c_2$. (2) $y = c_1 \log x + c_2$. (3) $2(y - b) = e^{x-a} + e^{-(x-a)}$. (4) $y = c_1 \log(1 + x) + \frac{1}{2} x - \frac{1}{4} x^2 + c_2$. **8.** (1) $y^2 = x^2 + c_1 x + c_2$. (2) $\log y = c_1 e^x + c_2 e^{-x}$. (3) $(x - c_1)^2 = c_2 (y^2 + c_2)$. (4) $y = \log \cos(c_1 - x) + c_2$.

Page 351. (1) $r = a \sin\theta$. (2) $xe^y = c(1 + x + y)$. (3) $c(2y^2 + 2xy - x^2)^{2\sqrt{3}} = \dfrac{(\sqrt{3}+1)x + 2y}{(1 - \sqrt{3})x + 2y}$. (4) $x^2 = 2cy + c^2$. (5) $y \sec x = \log(\sec x + \tan x) + c$.
(6) $3y = x^2(1 + x^2)^{\frac{3}{2}} + cx^2$. (7) $3x^2 + 4xy + 5y^2 + 5x + y = c$.
(8) $(x - 2c)y^2 = c^2 x$. (9) $y(x^2 + 1)^2 = \tan^{-1} x + c$. (10) $60y^3(x + 1)^2 = 10x^6 + 24x^5 + 15x^4 + c$. (11) $x = \dfrac{p}{\sqrt{1 - p^2}}(c + a \sin^{-1} p)$, $y = -ap + \dfrac{1}{\sqrt{1 - p^2}}(c + a \sin^{-1} p)$. (12) $x + c = a \log(p + \sqrt{1 + p^2})$, $y = a\sqrt{1 + p^2}$
(13) $y^2 = cx^2 - \dfrac{ch^2}{c+1}$. (14) $x = cxy + c^2$. (15) $y = \frac{3}{2}(p^2 + p) + \frac{3}{4} \log(2p - 1)$.
(16) $y(1 \pm \cos x) = c$. (17) $y^2 + (x + c)^2 = a^2$; $y^2 = a^2$. (18) $y = cx + \sqrt{b^2 + a^2 c^2}$; $b^2 x^2 + a^2 y^2 = a^2 b^2$. (19) $9(y + c)^2 = 4x(x - 3a)^2$; $x = 0$. (20) $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \sin(ax + \alpha)$. (21) $y = (c_1 e^x + c_2 e^{-x}) \cos x + (c_3 e^x + c_4 e^{-x}) \sin x$.
(22) $y = e^{2x}(c_1 + c_2 x) + c_3 e^{-x}$. (23) $y = c_1 x + c_2 x^{-1}$. (24) $y = \dfrac{c_1}{x} + x^{\frac{1}{2}} \left\{ c_2 \cos\left(\dfrac{\sqrt{3}}{2} \log x \right) + c_3 \sin\left(\dfrac{\sqrt{3}}{2} \log x \right) \right\}$. (25) $y = c_1(x + a)^2 + c_2(x + a)^3$.
(26) $(c_1 x + c_2)^2 + a = c_1 y^2$. (27) $3x = 2a^{\frac{1}{2}}(y^{\frac{1}{2}} - 2c_1)(y^{\frac{1}{2}} + c_1)^{\frac{1}{2}} + c_2$.
(28) $y = c_1 \log x + \frac{1}{2} x^2 + c_2$. (29) $e^{-ay} = c_1 x + c_2$.

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